CCFA-643422

Separation of fracture modes in functionally graded material beams

Masoud Tahani[*], Amir Mohammad Hosseini

Department of Mechanical Engineering, Faculty of Engineering, Ferdowsi University of Mashhad, Mashhad, Iran

Abstract

A functionally graded material (FGM) is a composite consisting of two or more phases, which is fabricated such that its composition varies in some spatial direction. The design is intended to take advantage of certain desirable features of each of the constituent phases. The knowledge of crack growth in FGMs is important in order to evaluate their integrity.

In this paper, a procedure of total potential energy release rate is developed for a general FGM beam, which is considered as a double cantilever beam. The mechanical properties vary as a general function of thickness coordinate. The procedure is based on a technical engineering theory for calculation of strain energy release rate and its separation into modes I and II of fracture. By choosing a suitable displacement field based on second-order shear-thickness deformation theory and using the principle of minimum total potential energy, the equilibrium equations are obtained along with the appropriate boundary conditions.

The total strain energy release rate is obtained by using the J-integral for the double cantilever beam (DCB) model. The path independent J-integral is calculated along the paths with variable distance from the stress singularity point and the convergence curve for J-integral is obtained. The distribution of stresses, strains and displacements are determined for some special loading conditions and the results are compared with the results of finite element method. This comparison shows good agreements between the results. The separation of J-integral into modes I and II is carried out by decomposition of stress and strain fields from point symmetric to the crack plane. The separation technique is applicable for both symmetric and unsymmetric geometry with respect to the crack plane. The results of the J-integral are compared with those obtained from other methods and it is found that there is good agreement between the results.

---

[*] Corresponding author. Tel: +98-511-876-3304; fax: +98-511-882-9541.
E-mail address: mtahani@ferdowsi.um.ac.ir (M. Tahani).

17

Separation of Fracture Modes in Functionally Graded Material Beams

[1]M. Tahani[*] and [2]A. Mohammad Hosseini

[1]Associate professor[*], [2]M.S. student, Department of Mechanical Engineering, Faculty of Engineering,
Ferdowsi University of Mashhad, Mashhad, Iran
[*](Tel: (511) 876-3304, e-mail: mtahani@ferdowsi.um.ac.ir)

Abstract

A procedure of total potential energy release rate is developed for a general FGM beam, which is considered as a double cantilever beam (DCB). The mechanical properties vary as a general function of thickness. The procedure is based on a technical engineering theory for calculation of strain energy release rate. By choosing a suitable displacement field based on second-order shear-thickness deformation theory and using the principle of minimum total potential energy, the equilibrium equations are obtained. The energy release rate is obtained by using the J integral for the DCB model. The distribution of displacements and stresses are determined for a special case and the results are compared with the results of finite element method. This comparison shows good agreement between the results.
The separation of J integral into modes I and II is carried out by decomposition of stress and strain fields from point symmetric to the crack plane. The separation technique is applicable for both symmetric and asymmetric geometry with respect to the crack plane. The results of J integral are compared with those of the other methods and present good agreement between the results.

Keywords: Functionally graded material; double cantilever beam; second-order shear-thickness deformation theory (SSDT); J integral; fracture modes

1. Introduction

Functionally graded materials (FGMs) possess properties that vary gradually with location within the material. As the use of FGMs increases, new methodologies have to be developed to characterize FGMs and also to design and analyze structural components made of these materials. The methods should be such that they can be incorporated into available methods with least amount of modifications. One such problem is that of determination of fracture parameters for FGMs. Fracture may occur as a result of accumulation of voids during processing, shock loading, or impact loading during service of the structure. Stress intensity factor and strain energy release rate are parameters to determine crack growth. The crack growth initiate, when the parameters reach to its critical limit. Then we can predict the fracture by determining these parameters. Both subjects of energy release rate and stress intensity factor have been studied by many researchers since the early 1960s. Most of them confined their research to homogeneous isotropic materials, some of them to homogeneous bimaterial or to orthotropic and very few to non-homogeneous materials [1]. By reviewing related papers on this subject it is seen that most of the methods are based on the finite element models. Analytical work on FGMs goes back as early as the late 1960s when soil was modeled as a nonhomogeneous material. Delale and Erdogan [2] analytically studied crack problem in an infinite plane where the elastic properties varied exponentially in the direction of the crack. Eischen [3] studied the crack-tip-singular behavior of the stress field in a nonhomogeneous infinite plane by using an eigenfunction expansion technique. Jain and Rousseau [4] studied crack tip stress field in FGMs with linearly varying properties using an elasticity solution. Anlas et al. [5] calculated stress intensity factor in FGMs.
Among the various methods presented, there appears to be no straightforward method for calculating stress

field and strain energy release rates under general loading conditions with a good degree of approximation. The main objective of the present study is to introduce an analytical model for fracture in FGM beams under general edge loading conditions, and to use this model for calculating strain energy release rate along with its separation into different modes of fracture.

2. Theoretical formulation

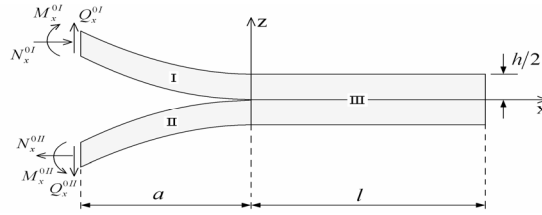The geometry of the structure and loads are shown in Figure 1.

Figure 1: Double cantilever beam, region I: $-a \leq x \leq 0$, $0 \leq z \leq h/2$, region II: $-a \leq x \leq 0$, $-h/2 \leq z \leq 0$, region III: $0 \leq x \leq l$, $-h/2 \leq z \leq h/2$

The geometry is divided into three regions (i.e., regions I, II, and III). The crack length is a and the total length of the beam is $a+l$. The mechanical properties vary in z-direction and are constant in x-direction. A second-order shear-thickness deformation theory is used for modeling the displacement field as follows:

$$u_1(x,y,z) = u(x) + z\psi_x(x) + z^2\eta_x(x)$$
$$u_2(x,y,z) = 0$$
$$u_3(x,y,z) = w(x) + z\psi_z(x)$$

(1)

where u_1, u_2, and u_3 are displacement components in x-, y-, and z-directions respectively. To develop a beam theory it is needed to assume that $\sigma_y=0$ [6]. Using the principle of minimum total potential energy, equilibrium equations can be shown to be:

$$\delta u: \quad \frac{dN_x}{dx} = 0, \qquad \delta w: \quad \frac{dQ_x}{dx} = 0, \qquad \delta\psi_x: \quad \frac{dM_x}{dx} - Q_x = 0$$

$$\delta\psi_z: \quad \frac{dR_x}{dx} - N_z = 0, \qquad \delta\eta_x: \quad \frac{dP_x}{dx} - 2R_x = 0$$

(2)

The moment and stress resultants are as follows:

$$\begin{Bmatrix} N_x \\ N_z \\ M_x \\ P_x \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{13} & B_{11} & D_{11} \\ A_{13} & A_{33} & B_{13} & D_{13} \\ B_{11} & B_{13} & D_{11} & E_{11} \\ D_{11} & D_{13} & E_{11} & F_{11} \end{bmatrix} \begin{Bmatrix} u' \\ \psi_z \\ \psi'_x \\ \eta'_x \end{Bmatrix}, \qquad \begin{Bmatrix} Q_x \\ R_x \end{Bmatrix} = \begin{bmatrix} A_{55} & B_{55} \\ B_{55} & D_{55} \end{bmatrix} \begin{Bmatrix} w' + \psi_x \\ \psi'_z + 2\eta_x \end{Bmatrix}$$

(3)

where the rigidity terms are defined as:

$$\left(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}\right) = \int_{z_L}^{z_u} C_{ij}\left(1, z, z^2, z^3, z^4\right)dz \qquad (i,j = 1,3)$$

$$\left(A_{55}, B_{55}, D_{55}\right) = k^2 \int_{z_L}^{z_u} C_{55}\left(1, z, z^2\right)dz \qquad (k^2 = 5/6)$$

(5)

There are five coupled second-order ordinary differential equations for each region which is solvable for any kind of boundary conditions. There are also five force boundary conditions at $x=-a$ in region I. Similarly there are five force boundary conditions for region II. By the assumption of a built in edge in region III at $x=l$, there will be five displacement boundary conditions. The displacement continuity conditions at $x=0$, where three regions are attached together, make ten displacement boundary conditions. Finally there will be five force continuity conditions at $x=0$.

3. Strain energy release rate

Crack growth can be described by the energy method. In the linear elastic fracture mechanics (LEFM), the strain energy release rate which is used as a criterion for determining crack growth, is defined as:

$$G = \frac{\partial W}{\partial A} - \frac{\partial U}{\partial A} \tag{6}$$

where W is the external work done on the body, U is the total strain energy, and A is the crack area. In LEFM, the strain energy release rate is identical to path independent J integral which is defined as:

$$J = \int_{\Gamma} \left(W dz - \sigma_{ij} n_j \frac{\partial u_i}{\partial x} d\Gamma \right) \tag{7}$$

where Γ is an arbitrary counterclockwise path, W is the strain energy density per unit volume, n_j's are the components of unit outward normal vector to the path, σ_{ij} is the stress tensor, and u_i's are the displacement components [7]. Upon substitution of $W=\sigma_{ij} \varepsilon_{ij}/2$, strains, and displacements into Equation (7) and integrating along an arbitrary path as seen in Figure 2, the J integral can be obtained:

$$J = J_1 \big|_{x=x_1} + J_2 \big|_{x=x_1} + J_3 \big|_{x=x_2} \tag{8}$$

where J_1, J_2, and J_3 are the J integrals along the path $x=x_1$ at region I, $x=x_1$ at region II, and $x=x_2$ at region III.
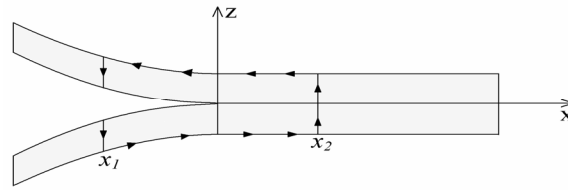

Figure 2: Path of integration

It is to be noted that since the traction vector vanishes on the horizontal divisions of the prescribed path, the J integral also vanishes on these divisions of the path. To this end, the integration path is summarized only to the vertical divisions [8].

4. Separation of mixed mode J integral to modes I and II

In the mixed mode fracture, the value of J can be decomposed into three components as follows:

$$J = J^{I} + J^{II} + J^{III} \tag{9}$$

where I, II, and III denote the three fracture modes. In the decomposition method the modes I, II, and III of J integral are directly obtained from modes I, II, and III stresses and displacements. To this end, the stress and strain products are obtained from points symmetric to the crack plane [9]. Then stress can be written as:

$$\sigma_u = (\sigma_u + \sigma_d)/2 + (\sigma_u - \sigma_d)/2, \quad \sigma_d = (\sigma_u + \sigma_d)/2 - (\sigma_u - \sigma_d)/2 \tag{10}$$

where σ_u and σ_d are the stress tensors up and down of the crack surface. The first term in Equations (10) is the symmetric part of stress which is concerned with mode I of fracture and the second one is the anti-symmetric part which is concerned with mode II of fracture.

5. Numerical results and discussions

In this section numerical study of an FGM beam will be made based on the mentioned theory. The FGM chosen for the study consists of Ti-6Al-4V (metal) and ZrO$_2$ (ceramic) with the following properties [10]:
Ti-6Al-4V: $E = 66.2$ GPa , $\upsilon=0.321$,   ZrO$_2$: $E = 117$ GPa , $\upsilon=0.321$
The properties change smoothly from Ti-6Al-4V on the bottom surface to ZrO$_2$ on the top surface of the beam. Without loss of generality, a linear distribution of mechanical properties is considered for the study. It is assumed that $h=5$ mm, $a=35$ mm, and $l=100$ mm. Here for brevity only one mode of loading is considered. The loading conditions of the example are as follows:

$$N_x^{0I} = N_x^{0II} = M_x^{0I} = M_x^{0II} = 0, \quad Q_x^{0I} = Q_x^{0II} = 10 \text{ N} \tag{11}$$

It must be noted that the accuracy of the results of this theory can be further enhanced by using higher-order terms in the expansion used for the displacement field or by using an improvement in the present theory. For the DCB specimen the actual distribution of u_3 on the thickness of the beam, require at least a third-order polynomial with respect to z-coordinate. On the other hand, in the present theory, a first-order polynomial can not be a good approximation for u_3. As a result ε_z and σ_z don't have enough accuracy. Then in the improved theory, the effect of ε_z on σ_x is neglected and the compliance matrix is considered as follows:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} 1/E & 0 & 0 \\ -v/E & 1/E & 0 \\ 0 & 0 & 2(v+1)/E \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_z \\ \sigma_{xz} \end{Bmatrix} \tag{12}$$

In this section the results obtained from SSDT and improved-SSDT (ISSDT) are also compared with FEM results of ANSYS. For the DCB specimen displacements and stresses at the middle plane of the beam (i.e., the crack plane, at $z=0$) are compared with FEM results in Figures 3-5. The J integral is computed along paths with various distances from crack tip which previously named as x_1 and x_2 and the results are presented in Figure 6a. It is seen that the convergence of J integral will occur in a path far away from the crack tip. Figure 6b presents the amounts of J integral and strain energy release rate versus crack length. For determining G by FEM data, at first the stiffness of the beam for each load is computed and then G is calculated using the following relation which is used for a structure with constant external loads [11]:

$$G = -\frac{\partial U}{\partial a} = -\frac{d}{da}\left(\frac{\Delta F(a)}{2}\right) = -\frac{1}{2}\Delta\frac{dF(a)}{da} = -\frac{1}{2}\Delta^2\frac{dk(a)}{da} \tag{13}$$

where Δ is the displacement of the load point and k is the stiffness. It is seen that fairly good agreements
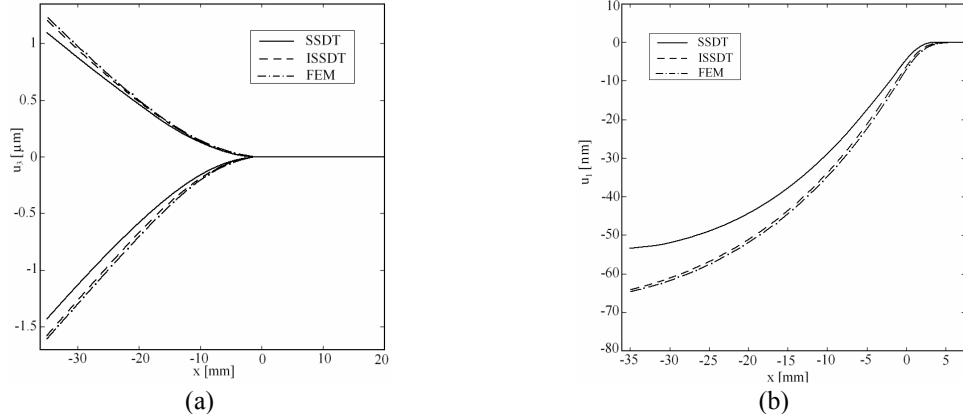
exist among the various results.

(a)   (b)

Figure 3: (a) Transverse deflection and (b) longitudinal stretching along specimen length at midplane

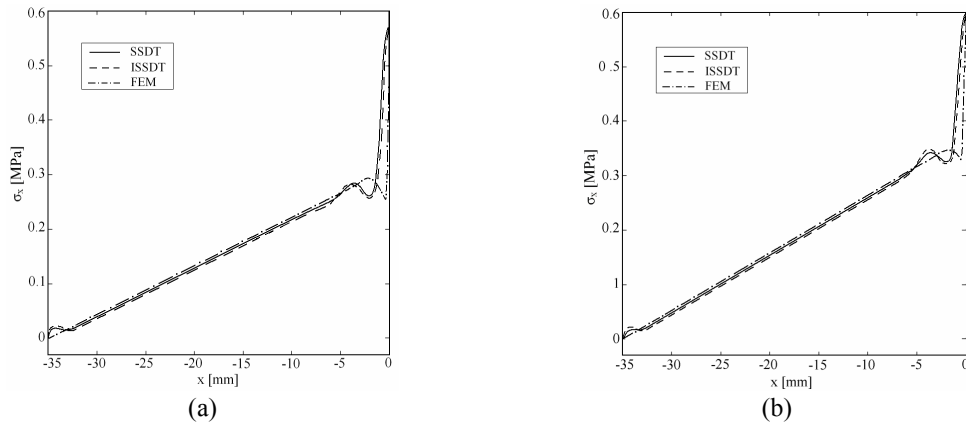(a)   (b)

Figure 4: Longitudinal normal stress σ_x along specimen length at midplane (a) in region I and
(b) in region II

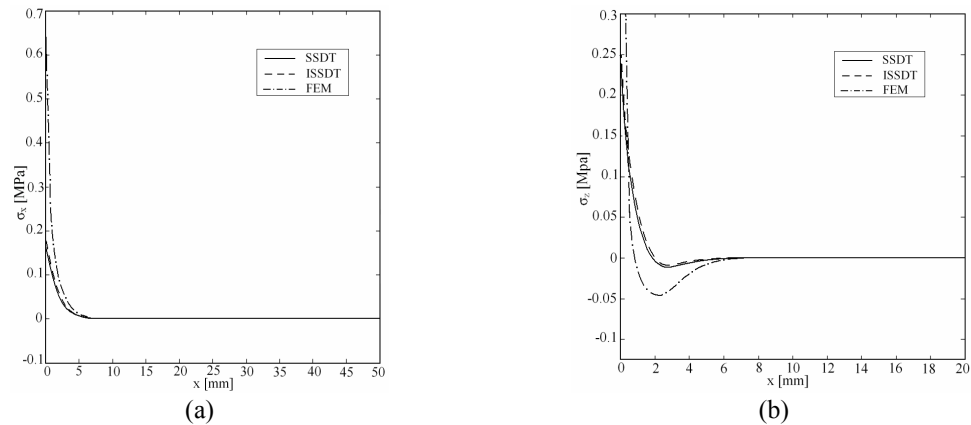(a)   (b)

Figure 5: (a) Longitudinal normal stress σ_x and (b) interlaminar normal stress σ_z along specimen length
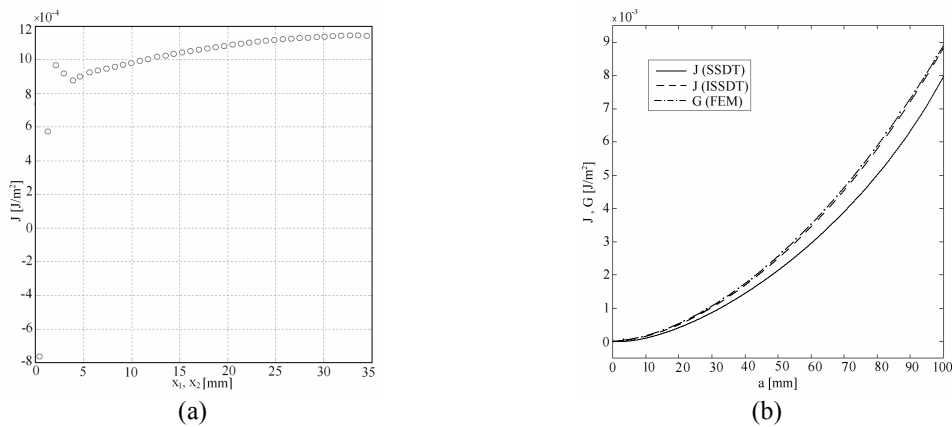at midplane

Figure 6: (a) Convergence of *J* integral and (b) *J* integral and *G* versus crack length

6. Conclusions

In this study, displacement and stresses in the cracked FGM beam are computed under general edge load conditions. The analysis treats the cracked beams as three beams joined together at the crack tip. A displacement field consistent with the physical deformation field is used in the minimum total potential energy principle for obtaining the equilibrium equations. After obtaining analytical solutions for the equilibrium equations, the strain energy release rate is obtained by using the definition of the *J* integral. The strain energy release rate is decomposed into fracture modes. The results are compared with FEM results. It is found that strain energy release rate increases by increasing crack length.

References

[1] Sheinman, I. and Kardomateas, G.A. (1997), *International Journal of solids and structures* **34(4)**, 451-459.
[2] Delale, F. and Erdogan, F. (1988), *International Journal of Engineering Science* **26**, 559-568.
[3] Sosa, H.A. and Eischen, J.W. (1986), *Engineering Fracture Mechanics* **25**, 451-462.
[4] Jin, N. and Rousseau, C.E. (2004), *Theoretical and Applied Fracture Mechanics* **42**, 155-170.
[5] Anlas, G., Santare, M.H., and Lambros, J. (2000), *International Journal of Fracture* **104**, 131-143.
[6] Tahani, M. (2007), *Composite Structures* **79**, 535-547.
[7] Greco, F. and Lonetti, P. (2002), *International Journal of Solids and Structures* **39**, 2435-2463.
[8] Sorensen, B.F. and Kirkegaard, P. (2006), *Engineering Fracture Mechanics* **73**, 2642–2661.
[9] Hamed, M.A. and Nosier, A. (2006), *Materials and Design* **27**,900-910.
[10] Bahtui, A. and Eslami, M.R. (2005), *Mechanics Research Communications* **27**,301-312.
[11] Wang, C.H. (1996), Introduction to Fracture Mechanics, Melbourne: DSTO Aeronautical and Maritime Research Laboratory.