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Analysis of interlaminar stresses in general cross-ply composite plates subjected to free vibration

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Abstract

It is well known that a mismatch in elastic properties between adjacent plies of composite laminates causes interlaminar stress concentrations near the edges of a laminate where we have material discontinuity. These stresses can initiate heterogeneous damage in the forms of delamination and transverse cracking and may cause the damage to propagate to a substantial region of the laminate, resulting in a significant loss of strength and stiffness. In the edge zone of the laminate, it has been shown that the state of stresses is three-dimensional in nature and the classical lamination theory cannot be employed.

In this paper, within the displacement field of the first-order shear deformation theory, free vibrations of rectangular cross-ply composite plates are studied and the natural frequencies and dynamic interlaminar stresses are obtained. In the theoretical formulations the effects of all the rotational inertia terms are considered. Also the change in the plate thickness is taken into account due to its important role in the edge effects. The equations of motion are derived by using Hamilton's principle. It is assumed that the plates have two simply supported opposite edges and the remaining boundary conditions are arbitrary. The obtained equations are solved analytically using the state-space approach for the case of free vibration. First the natural frequencies and the mode shapes are obtained and then the interlaminar stresses are determined by integrating the three-dimensional local equations of motion and utilizing given boundary conditions. The accuracy and effectiveness of the present theory in describing the localized three-dimensional effects are demonstrated by comparing the results of the first-order theory with those obtained from the finite element method. It is found that the theory can predict the natural frequencies and the dynamic interlaminar stresses. The theoretical analyses and results are of certain significant in determining dynamic interlaminar stresses in the case of transient loadings and also in practical engineering applications.

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Analysis of Interlaminar Stresses in General Cross-Ply Composite Plates Subjected to Free Vibration

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Abstract

In this paper, within the displacement field of the first-order shear deformation theory, free vibrations of cross-ply composite rectangular plates are studied and the natural frequencies and dynamic interlaminar stresses are obtained. In the theoretical formulations the effects of all the rotational inertia terms are considered. Also the change in the plate thickness is taken into account due to its important role in the edge effects. The obtained equations are solved analytically using the state-space approach for the case of free vibration. The accuracy and effectiveness of the present theory are demonstrated by comparing the results of the first-order theory with those obtained from the finite element method.

Keywords: Composite materials; cross-ply laminates; free vibration; first-order shear deformation theory; interlaminar stresses

1. Introduction

Laminated composite plates are being increasingly used in aeronautical and aerospace industry as well as in other fields of modern technology. As an efficient use a good understanding of their structural and dynamical behavior and also a verified consideration of the deformation characteristics, stress distribution, natural frequencies, and buckling loads under various load conditions are expected. Several representative researchers had surveyed the development of the study on vibrations of composite laminated plates. It is possible to find an analytical solution for the dynamic and static behavior of cross-ply laminated plates subjected to simply supported boundary conditions at their opposite two edges and different boundary conditions at the remaining ones. For these types of problems, Reddy and Khedir [1] developed a Levy-type solution in conjunction with the state-space technique on the basis of a parabolic shear deformation theory. Aydogdu and Timarci [2] concerned the vibration analysis of cross-ply laminated square plates subjected to different sets of boundary conditions. Kant and Swaminathan [3] applied a higher-order refined theory to free vibration of composite laminates and sandwich plates with simply supported boundary conditions. Using the modified complementary energy principle, Afshari and Widera [4] developed a series of plate elements for free vibration of composite laminated plates by the Mindlin thin plate theory.

One of the main causes of failure of composite laminated structures is delamination damage, which is significantly derived from interlaminar stresses. Previous researches were mainly limited to the interlaminar stress distributions in laminated structures under static loads, while response histories and distribution of dynamic interlaminar stresses were seldom mentioned. Jane and Hong [5] determined the interlaminar stresses in a laminated rectangular orthotropic plate with four sides simply supported during free vibration by using the integration method involving the dynamic inertia terms and displacements. Wang et al. [6] studied the response histories and distribution of dynamic interlaminar stresses in laminated plates with simple and fixed supports, subjected to free vibration and thermal load but the change in the plate thickness which has significant effect on the dominant interlaminar stresses is neglected in these studies.

In this paper, an analytical formulation is developed for the free vibration analysis of general cross-ply laminated composite plates within the framework of the first-order shear deformation theory. The solutions are obtained analytically and the natural frequencies and dynamic interlaminar stresses are determined. In the theoretical formulations the effects of all the rotational inertia terms are considered. Also the change in the plate thickness is taken into account due to its important role in the edge effects.

2. Mathematical formulations

It is intended here to determine the interlaminar stresses in a general cross-ply laminate subjected to free vibration. The geometry of the laminate is shown in Figure 1. The formulation is restricted to linear elastic material behavior and small strain and displacements.

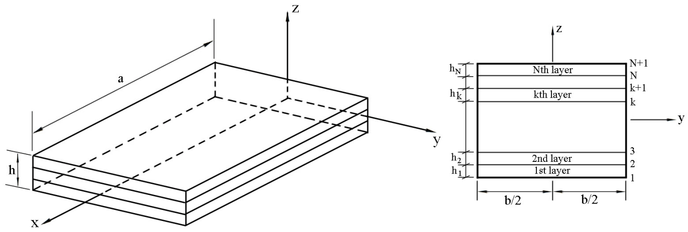


Figure 1: Laminate geometry and coordinate system.

2.1. Displacement field and strains

In order to approximate the three-dimensional elasticity problem to a two-dimensional plate problem, the displacement components $u_1(x,y,z,t)$, $u_2(x,y,z,t)$, and $u_3(x,y,z,t)$ at any point in the plate space are expanded in a Taylor's series in terms of thickness coordinate. The displacement field of the first-order shear deformation theory (FSDT) may be assumed as

$$u_1(x,y,z,t) = u(x,y,t) + z\psi_x(x,y,t), \qquad u_2(x,y,z,t) = v(x,y,t) + z\psi_y(x,y,t)$$
$$u_3(x,y,z,t) = w(x,y,t) + z\psi_z(x,y,t) \tag{1}$$

It is to be noted that the elasticity solution indicates that the transverse shear stress vary parabolically through the plate thickness. This requires the use of shear correction factors. Furthermore, the transverse normal strain may vary nonlinearly through the plate thickness.

In Eqs. (1) u_1, u_2, and u_3 are displacement components in the *x*, *y*, and *z* directions respectively, *u* and *v* are the in-plane displacements and *w* is the transverse displacement of a point (*x*,*y*) on the middle plane. The functions ψ_x and ψ_y are the rotations of a normal transverse to the middle plane about *y*- and *x*-axes, respectively, and ψ_z is the thickness variability parameter.

By substitution of the displacement field in (1) into the strain-displacement relations [7] of elasticity, the following results will be obtained

$$\varepsilon_x = \frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \psi_x}{\partial x}, \quad \varepsilon_y = \frac{\partial u_2}{\partial y} = \frac{\partial v}{\partial y} + z\frac{\partial \psi_y}{\partial y}, \quad \varepsilon_z = \frac{\partial u_3}{\partial z} = \psi_z$$
$$\gamma_{yz} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \psi_y + \frac{\partial w}{\partial y} + z\frac{\partial \psi_z}{\partial y}, \quad \gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \psi_x + \frac{\partial w}{\partial x} + z\frac{\partial \psi_z}{\partial x}$$
$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + z\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) \tag{2}$$

2.2 Equations of motion

The displacement field in Eqs. (1) can be used to drive the equations of motion by means of Hamilton's principal [8]. The equations of motion (Euler-Lagrange equations) are as follows

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_1\ddot{u} + I_2\ddot{\psi}_x, \quad \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = I_2\ddot{u} + I_3\ddot{\psi}_x, \quad \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x,y,t) = I_1\ddot{w} + I_2\ddot{\psi}_z$$
$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_1\ddot{v} + I_2\ddot{\psi}_y, \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = I_2\ddot{v} + I_3\ddot{\psi}_y, \quad \frac{\partial R_x}{\partial x} + \frac{\partial R_{xy}}{\partial y} - N_z = I_2\ddot{w} + I_3\ddot{\psi}_z \tag{3}$$

with

$$\left(N_x,N_y,N_z,N_{xy}\right)=\int_{-h/2}^{h/2}\left(\sigma_x,\sigma_y,\sigma_z,\sigma_{xy}\right)dz\,,\qquad \left(M_x,M_y,M_{xy}\right)=\int_{-h/2}^{h/2}\left(\sigma_x,\sigma_y,\sigma_{xy}\right)zdz$$
$$\left(Q_x,Q_y\right)=\int_{-h/2}^{h/2}\left(\sigma_{xz},\sigma_{yz}\right)dz\,,\quad \left(R_x,R_y\right)=\int_{-h/2}^{h/2}\left(\sigma_{xz},\sigma_{yz}\right)zdz\,,\quad \left(I_1,I_2,I_3\right)=\int_{-h/2}^{h/2}\rho\left(1,z,z^2\right)dz \tag{4}$$

where N_y, N_z, N_{xy}, M_x, M_y, and M_{xy} are the stress resultants and Q_x and Q_y are the transverse shear force resultants. Also I_1, I_2, and I_3 are the corresponding inertia terms.

2.3. Constitutive equations

The linear constitutive relations for the *k*th orthotropic lamina, with fiber orientations of 0° and 90° only, with respect to the laminate coordinate axes (see Figure 1) are given by [9]

$$\{\sigma\}^{(k)}=[\bar{C}]^{(k)}\{\varepsilon\}^{(k)} \tag{5}$$

Here, the matrix $[\bar{C}]^{(k)}$ is called the off-axis stiffness matrix. Upon substitution of Eqs. (2) into Eq. (5) and the subsequent results into Eqs. (4) and (3), the displacement equations of motion will be obtained.

3. Analytical solutions

Here the exact solution of Eqs. (3) for cross-ply rectangular plates are considered. By omitting the applied transverse load, the equations of free vibration will be obtained. The process of solving the governing differential equations consists of *Levy's formulations* [9]. Levy's solution exists when at least two opposite edges of the plate have simple supports. The remaining edges may have simple, clamped or free boundary conditions. To this end, it is assumed here that the edges of the plate at *x*=0 and *x*=*a* have the following boundary conditions

$$N_x=v=M_x=\psi_y=\psi_z=w=0 \tag{6}$$

It is noted that the boundary conditions in (6) will identically be satisfied if the following expressions for the displacement components are assumed

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}(y)\cos\alpha_m x e^{i\omega_n t}\,,\qquad v(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn}(y)\sin\alpha_m x e^{i\omega_n t}$$
$$w(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}(y)\sin\alpha_m x e^{i\omega_n t}\,,\qquad \psi_x(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_{xmn}(y)\cos\alpha_m x e^{i\omega_n t}$$
$$\psi_y(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_{ymn}(y)\sin\alpha_m x e^{i\omega_n t}\,,\qquad \psi_z(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_{zmn}(y)\sin\alpha_m x e^{i\omega_n t} \tag{7}$$

where $\alpha_m=m\pi/a$ with *m* being the Fourier integer. Upon Substitution of Eqs. (7) into the governing equations of motion, the set of partial differential equations are transformed to a set of ordinary differential equations. An alternative method of solving of the obtained equations is provided by the state-space approach. By the aid of this approach, the six ordinary second-order differential equations can be expressed to the following system of twelve first-order ordinary equations

$$\{Y'\}=[T]\{Y\} \tag{8}$$

The general solution of Eq. (8) is given by

$$\{Y_{mn}\} = [E_{mn}][Q_{mn}]\{K_{mn}\} \tag{9}$$

where

$$[Q_{mn}] = \begin{bmatrix} e^{\lambda_{1mn}y} & 0 & \ldots & 0 \\ 0 & e^{\lambda_{2mn}y} & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & & e^{\lambda_{12mn}y} \end{bmatrix} \tag{10}$$

In Eq. (9) $[E_{mn}]$ and λ_{imn} (i = 1,2,...,12) are, respectively, the matrix of eigenvectors and eigenvalues of the coefficient matrix [*T*] which, in general, must be regarded to have complex values. Also $\{K_{mn}\}$ being 12*mn* arbitrary unknown constants of integration to be found by imposing the boundary conditions at $y = \pm b/2$. Three different boundary conditions in the FSDT that may be exist at $y = \pm b/2$ are as follows

$$\begin{aligned} &\text{Free (F):} && N_{xy} = N_y = Q_y = M_{xy} = M_y = R_y = 0 \\ &\text{Simple support (S):} && N_y = u = M_y = \psi_x = \psi_z = w = 0 \\ &\text{Clamped (C):} && u = v = w = \psi_x = \psi_y = \psi_z = 0 \end{aligned} \tag{11}$$

Imposing the boundary conditions at $y = \pm b/2$ yields

$$[M_{mn}]\{K_{mn}\} = \{0\} \tag{12}$$

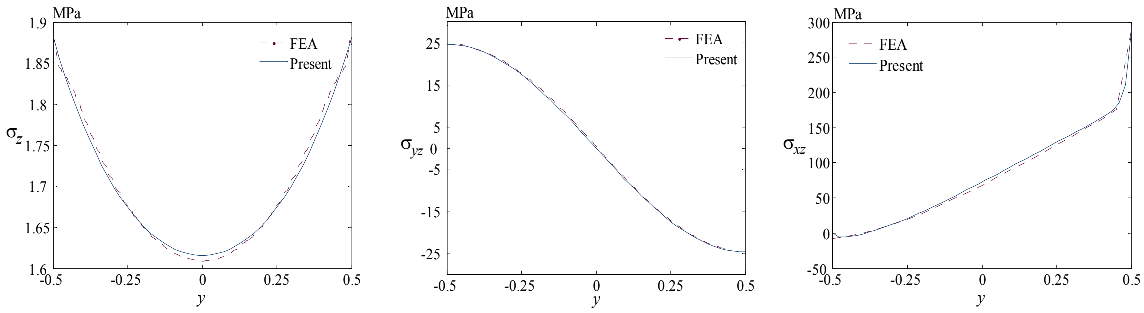
For nontrivial solution, the determinate of the coefficient matrix in (12) should be zero. The roots of this equation for each (*m*,*n*) are the natural frequencies. Next, the mode shapes of vibration for each natural frequency may be obtained from Eqs. (12) and (9).

4. Numerical results and discussion

In what follows several numerical examples are presented for symmetric and antisymmetric cross-ply laminates subjected to free vibrations. The laminates have length *a*, width *b*, and thickness *h* (see Figure 1). Each lamina is assumed to be of the same thickness $h_k = h/N$, where *N* is the number of laminae. The non-dimensional natural frequencies $\overline{\omega}$ of general cross-ply composite plates with simple supports are considered for comparison. The non-dimensional natural frequencies are tabulated in Table 1 for antisymmetric cross-ply square laminates with two, four, six, and ten layers. The orthotropic material properties of the individual layers are assumed to be $E_1/E_2 =$ open, $E_2 = E_3$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$ (Material I).
Figure 2 shows the distributions of various in-plane and out-of-plane stresses through the *y* direction in a [0/90/0] laminate at $x = a/4$, with different boundary conditions. All stress distributions compared with the finite element analysis (FEA) and excellent agreements between the FSDT and FEA are found. The orthotropic material properties of individual layers in this example are $E_1/E_2 = 10$, $E_2 = 15500$ MPa, $G_{12} = G_{13} = G_{23} = 0.28E_2$, $\nu_{12} = \nu_{23} = \nu_{13} = 0.248$, $\rho = 1380$ kg/m^3 (Material II).
Figure 3 illustrates the variations of interlaminar shear stress σ_{yz} through the *y* direction of a [90/0/90] SFSF laminate at $x = a/4$ for material I, *a*/*b*=1 and $E_1/E_2 = 40$. It is seen that the magnitude of σ_{yz} is naturally increasing when approaching the free edge. Figure 4 presents the variations of interlaminar shear stress σ_{xz} through the *y* direction of a [0/90/0] SSSF laminate at $x = a/4$ and various values of E_1/E_2 for martial I, *a*/*b*=1 and *a*/*h*=10. The magnitude of σ_{xz} is naturally increasing when approaching the free edge. It is found that the magnitudes of σ_{yz} and σ_{xz} are increased in the boundary layer region and exhibit singular behaviour near the free edge.

Table 1: Non-dimensional fundamental frequencies $\bar{\omega} = \omega(a^2/h)\sqrt{\rho/E_2}$ for a square simply supported antisymmetric cross-ply laminated plates with $a/h = 5$

Laminate	Source	E_1/E_2				
		3	10	20	30	40
[0/90]	Present	6.3609	7.0426	7.7850	8.3855	8.8873
	Noor [11]	6.2578	6.9845	7.6745	8.1763	8.5625
	Reddy [10]	6.2169	6.9887	7.8210	8.5050	9.0871
	Kant-Swaminathan [4]	6.2336	6.9363	7.6883	8.2570	8.7097
$[0/90]_2$	Present	6.6530	8.3034	9.7345	10.6506	11.2930
	Noor [11]	6.5455	8.1445	9.4055	10.1650	10.6798
	Reddy [10]	6.5008	8.1954	9.6265	10.5348	11.1716
	Kant-Swaminathan [4]	6.4319	8.1010	9.4338	10.2463	10.7993
$[0/90]_3$	Present	6.7051	8.4939	9.9853	10.9106	11.5461
	Noor [11]	6.6100	8.4143	9.8398	10.6958	11.2728
	Reddy [10]	6.5552	8.4041	9.9175	10.8542	11.5007
	Kant-Swaminathan [4]	6.4319	8.3372	9.8012	10.6853	11.2838
$[0/90]_5$	Present	6.7315	8.5873	10.1048	11.0323	11.6630
	Noor [11]	6.6458	8.5625	10.0843	11.0027	11.6245
	Reddy [10]	6.5842	8.5126	10.0674	11.0197	11.6730
	Kant-Swaminathan [4]	6.5177	8.4680	10.0107	10.9445	11.5789



(a) (b) (c)

Figure 2: Distributions of (a) interlaminar normal stress σ_z of a SFSF laminate (b) interlaminar shear stress σ_{yz} of a fully simply supported laminate and (c) interlaminar shear stress σ_{xz} of a SCSF laminate through the y direction.

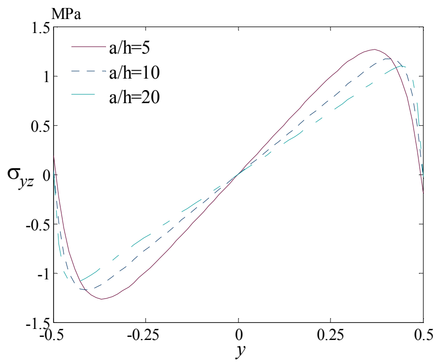


Figure 3: Distributions of interlaminar shear stress σ_{yz} through the y direction of a [90/0/90] SFSF laminate

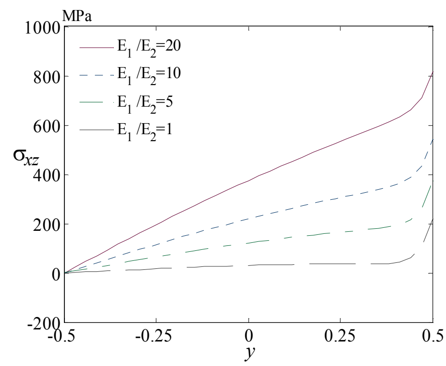


Figure 4: Distributions of interlaminar shear stress σ_{xz} through the y direction of a [90/0/90] SSSF laminate

Typical distributions of interlaminar normal stress σ_z through y direction for one, four, six, and eight layers antisymmetric cross-ply SCSC laminate at $x = a/4$ are shown in Figure 5. It is noted that the magnitude of σ_z is increased with increasing of number of laminae (*N*) while thickness of laminate (*h*) is constant. Distributions of in-plane normal stress σ_y through the y direction at $x = a/4$ for third and fourth vibration modes of a fully simply supported three-layer symmetric cross-ply laminate are shown in Figure 6. In Figures 5 and 6 material properties are the same as material I while $E_1 / E_2 = 20$, *a*/*b*=1 and *a*/*h*=10.

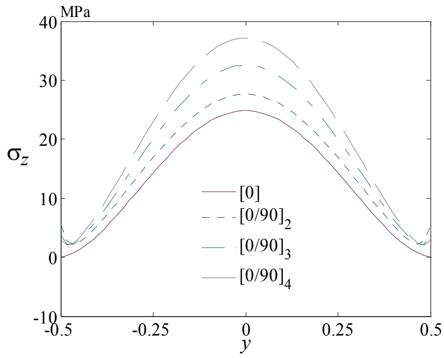


Figure 5: Distributions of interlaminar normal stress σ_z through the *y* direction of SCSC antisymmetric cross-ply laminates

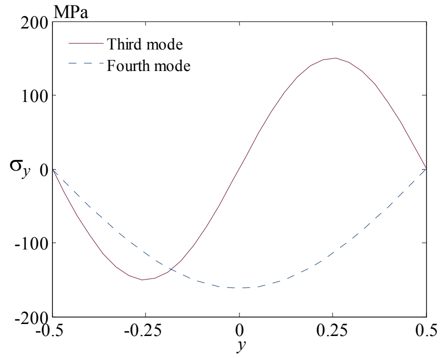


Figure 6: Distributions of in-plane normal stress σ_y through the *y* direction of a fully simply supported [90/0/90] laminate

5. Conclusions

In this paper, an analytical method is applied to calculate the response and distribution of dynamic interlaminar stresses in composite laminated plates with two opposite simply supported edges, subjected to free vibration. Within the framework of the first-order shear deformation theory and with considering all of the inertia terms and the transverse normal strain the equations of motion are obtained and a generalized Levy-type solution are developed. Numerical results show that dynamic interlaminar stresses are mainly determined by the vibration modes. It is found that the present results have excellent agreements with those obtained by using finite element method. These close agreements verify the accuracy of the first-order shear-thickness theory which is used in this case study.

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