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A STUDY OF PIEZOELECTRIC LAMINATED COMPOSITE BEAMS SUBJECTED TO ELECTRO-MECHANICAL LOADINGS

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ABSTRACT

In this paper, analytical solutions of displacements and stresses in beams with integrated sensors and actuators are obtained based on a new and the existing first-order shear deformation beam theories. It is assumed that the beams are subjected to both mechanical and electrical loadings. The new theory is simple and straightforward and also free from the inversion of certain matrices which can be an inconvenience as far as developing more advanced laminated beam theories are concerned. The procedure is general and, therefore, can be used to develop higher-order and layerwise theories. For the assessment of the accuracy of this theory, analytical solutions are obtained and compared with those of existing first-order beam theory. It is found that the new beam theory can predict accurately displacements and stresses in the beams.

1. INTRODUCTION

The fiber-reinforced composite materials play an important role in modern industry for its high strength-to-weight ratio. When piezoelectric materials, which can be used for both sensors and actuators, are bonded on the top/bottom surface or embedded in composite structures, their performance could be effectively enhanced.

In the last two decades, the subject area of smart/intelligent materials and structures has experienced tremendous growth in terms of research and development. One reason for this activity is that it may be possible to create certain types of structures and systems capable of adapting to or correcting for changing operating conditions.

Piezoelectric elements can be used as sensors or actuators in static applications such as torsion of helicopters blades, deflection of missiles fins, airfoil shape changes, or in dynamic applications such as structural vibration and acoustical generated noise. The design of such active systems requires accurate models of the electro-mechanical interaction between the structure and piezoelectric sensors or actuators.

An analytical model of piezolaminated composite beams has been treated by Abramovich and Livshits [1] and Abramovich [2]. In the model, which is based on the linear piezoelectric theory and pin-force model, the piezoelectric elements are usually bonded to the top and bottom of the structure or are embedded in the composite beam as continues layers. Applied voltage to the actuators, may induce in-plane extension, bending or both in-plane and bending deflections. For symmetric laminates, applying the same in-phase voltage to both actuators will produce in-plane deformation, whereas equal out-of-phase voltage will produce pure bending. Both in-plane and bending deformations will be induced when different voltages are applied to each actuator or when a single actuator is bonded to the structure.

Rammerstorfer [3] proved that it is possible to increase the first natural frequency and buckling load of plates by application of optimal fields of initial stresses. Almeida [4] suggested using of piezoelectric actuators for the stress stiffening effect of laminated composite beams and plates. Waisman and Abramovich [5] studied the stiffening effects of a smart piezolaminated composite beam. They solved numerically the three coupled equations of motion of a general non-symmetric piezolaminated composite beam subjected to axial and lateral tractions and various boundary conditions to obtain the natural frequencies and mode shapes.

Gaudenzi et al. [6] simulated the problem of the attenuation of the vibration effects in active cantilever beams by two strategies, position and velocity control with both numerical and experimental methods. Huang and Sun [7] modeled composite beams with bonded or embedded piezoelectric sensors and actuators to demonstrate the dynamic responses. Tong and Luo [8] and Luo and Tong [9] presented exact dynamic solutions to smart beams with a partially bonded piezoelectric patch. Based on the exact solutions, they obtained frequency spectra, natural frequencies, normal mode shapes, and harmonic responses of the shear and peel stresses for the PZT actuator. Vel et al. [10] obtained an analytical solution for the cylindrical bending vibrations of linear piezoelectric laminated plates by extending the Stroh formalism to the generalized plane strain vibrations of piezoelectric materials.

Pan and Heyliger [11] derived analytical solutions for the cylindrical bending of multilayered, linear, and anisotropic magneto-electroelastic plates under simple-supported edge conditions. Aldraihem and Khdeir [12] presented exact analytical solutions for deflection of beams with *n* actuators of shear piezoelectric.

As far as the development of a laminated beam theory is concerned, two different approaches are adopted in the literature. In the first approach the lateral (the *y*-direction) displacement of the beam is simply neglected. This way, the couplings between in-plane shearing and stretching and between bending and twisting are ignored. Such a theory is often used for isotropic beams and cross-ply laminated beams. In fact, this theory is for the cylindrical bending of laminated plates and not for the bending of laminated beams [13]. In the second approach a laminated beam theory is developed from an existing laminated plate theory. To this end, the stress (force) and moment resultants of the beam theory are obtained by ignoring certain stress and moment resultants in the constitutive law of the laminated plates. This way the characteristic couplings, mentioned earlier, are not lost in the beam theory. The process, however, demands the inversion of certain matrices which can be an inconvenience as far as developing more advanced laminated beam theories are concerned.

It is the intention of the present work to develop a new first-order shear deformation laminated beam theory to overcome the shortcomings present in the two approaches discussed above. That is, the displacement field will be modified so that the constitutive law of a laminated beam can be obtained in a straightforward manner as in most laminated plate and shell theories. The resulting equilibrium equations will be valid for generally laminated beams.

2. THEORETICAL FORMULATIONS

In what follows two first-order shear deformation laminated beam theories for the analysis of laminated beams with piezoelectric layers subjected to mechanical and/or electrical loadings will be derived. First, a first-order shear deformation laminated plate theory (FSDPT) will be used to derive first-order shear deformation beam theory 1 (FSDBT1). Next, a new first-order shear deformation beam theory (FSDBT2) will be developed.

2.1. FSDBT1

Consider a rectangular ($L \times b$) laminated composite plate of total thickness *h* with *N* layers (see Fig. 1). In the first-order shear deformation plate theory it is assumed that:

$$
\begin{aligned}
u_1(x,y,z) &= u_0(x,y) + z\psi_x(x,y) \\
u_2(x,y,z) &= v_0(x,y) + z\psi_y(x,y) \\
u_3(x,y,z) &= w(x,y)
\end{aligned}
\tag{1}
$$

where u_0, v_0, and *w* denote the displacements of a point on the middle plane of the plate (z=0). Also ψ_x and ψ_y are unknown functions which denote rotations of a cross-section about *y* and *x* axes, respectively. Upon substitution of Eqs. (1) into the linear strain-displacement relations of elasticity, the following results will be obtained:

$$
\begin{aligned}
&\varepsilon_x = \varepsilon_x^0 + z\kappa_x, \quad \varepsilon_y = \varepsilon_y^0 + z\kappa_y, \quad \varepsilon_z = 0 \\
&\gamma_{yz} = \gamma_{yz}^0, \quad \gamma_{xz} = \gamma_{xz}^0, \quad \gamma_{xy} = \gamma_{xy}^0 + z\kappa_{xy}
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
&\varepsilon_x^0 = \frac{\partial u_0}{\partial x}, \quad \kappa_x = \frac{\partial \psi_x}{\partial x}, \quad \varepsilon_y^0 = \frac{\partial v_0}{\partial y}, \quad \kappa_y = \frac{\partial \psi_y}{\partial y} \\
&\gamma_{yz}^0 = \psi_y + \frac{\partial w}{\partial y}, \quad \gamma_{xz}^0 = \psi_x + \frac{\partial w}{\partial x} \\
&\gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}, \quad \kappa_{xy} = \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}
\end{aligned}
\tag{3}
$$

Using the principle of minimum total potential energy the equilibrium equations can be shown to be:

$$
\delta u_0 : \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0
$$

$$\delta v_0 \ : \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0$$

$$\delta \psi_x : \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

$$\delta \psi_y : \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0$$

$$\delta w \ : \ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x,y) = 0 \qquad (4)$$

where $q(x,y)$ is the transverse load that is applied on the top surface (z=-h/2) of the laminate. Also the boundary conditions consist of specifying the following quantities at the edges of the plate:
at x=0, L;

Geometric (Essential)	Force (Natural)	
u_0	N_x	
v_0	N_{xy}	
w	Q_x	(5a)
ψ_x	M_x	
ψ_y	M_{xy}	

and at $y = \pm b/2$;

Geometric (Essential)	Force (Natural)	
u_0	N_{xy}	
v_0	N_y	
w	Q_y	(5b)
ψ_x	M_{xy}	
ψ_y	M_y	

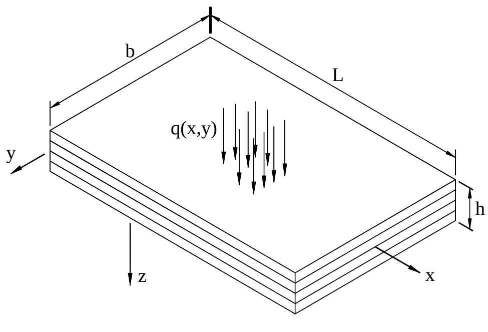

Figure 1. Geometry of piezoelectric laminated composite plate

In Eq. (4) the force and moment resultants are defined as:

$$\left(N_x, N_y, N_{xy}\right) = \int_{-h/2}^{h/2} \left(\sigma_x, \sigma_y, \sigma_{xy}\right) dz$$
$$\left(M_x, M_y, M_{xy}\right) = \int_{-h/2}^{h/2} \left(\sigma_x, \sigma_y, \sigma_{xy}\right) zdz \qquad (6)$$
$$\left(Q_x, Q_y\right) = \int_{-h/2}^{h/2} \left(\sigma_{xz}, \sigma_{yz}\right) dz$$

The linear constitutive relations for the kth orthotropic (piezoelectric) lamina in the laminate coordinates (x,y,z) are given as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^{(k)} - \begin{bmatrix} 0 & 0 & \bar{e}_{31} \\ 0 & 0 & \bar{e}_{32} \\ 0 & 0 & \bar{e}_{36} \end{bmatrix}^{(k)} \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}^{(k)} \qquad (7a)$$

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^{(k)} + \begin{bmatrix} \bar{e}_{14} & \bar{e}_{24} & 0 \\ \bar{e}_{15} & \bar{e}_{25} & 0 \end{bmatrix}^{(k)} \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}^{(k)} \qquad (7b)$$

where $\bar{Q}_{ij}^{(k)}$ are the transformed reduced plane-stress stiffnesses, $\bar{C}_{ij}^{(k)}$ are the transformed stiffnesses, $\bar{e}_{ij}^{(k)}$ are the transformed piezoelectric moduli of the kth lamina, and ($E_x^{(k)}, E_y^{(k)}, E_z^{(k)}$) are the components of electic field (for more complete descriptions of these terms see [14]). For layers other than piezoelectric layers, the part containing the piezoelectric moduli $\bar{e}_{ij}^{(k)}$ should be omitted. The piezoelectric stiffnesses $\bar{e}_{ij}^{(k)}$ are known in terms of the dielectric constants and elastic stiffnesses as:

$$\begin{bmatrix} 0 & 0 & \bar{e}_{31} \\ 0 & 0 & \bar{e}_{32} \\ 0 & 0 & \bar{e}_{36} \end{bmatrix}^{(k)} = \begin{bmatrix} 0 & 0 & \bar{d}_{31} \\ 0 & 0 & \bar{d}_{32} \\ 0 & 0 & \bar{d}_{36} \end{bmatrix}^{(k)} \cdot \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \qquad (8)$$

Upon substitution of Eqs. (2) into Eqs. (7a) and (7b) and the subsequent results into Eqs. (6), the stress resultants are obtained which can be presented as follows:

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^0\} \\ \{\kappa\} \end{Bmatrix} - \begin{Bmatrix} \{N^P\} \\ \{M^P\} \end{Bmatrix} \quad (9a)$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = k^2 \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma^0_{yz} \\ \gamma^0_{xz} \end{Bmatrix} - \begin{Bmatrix} Q^P_y \\ Q^P_x \end{Bmatrix} \quad (9b)$$

where k^2(=5/6) is the shear correction factor, {N} is the membrane force vector of the mid-plane, {M} is the bending moment vector and {N^P} and {M^P} are the membrane force and bending moment vectors caused by the electric field, respectively. That is,

$$\begin{aligned} \{N\} &= [N_x, N_y, N_{xy}]^T \\ \{M\} &= [M_x, M_y, M_{xy}]^T \\ \{N^P\} &= [N^P_x, N^P_y, N^P_{xy}]^T \\ \{M^P\} &= [M^P_x, M^P_y, M^P_{xy}]^T \end{aligned} \quad (10)$$

with

$$\begin{aligned} (N^P_x, N^P_y, N^P_{xy}) &= \int_{-h/2}^{h/2} (\bar{e}_{31}, \bar{e}_{32}, \bar{e}_{36}) E_z dz \\ &= \sum_{k=1}^{Na} \int_{z_k}^{z_{k+1}} (\bar{e}^{(k)}_{31}, \bar{e}^{(k)}_{32}, \bar{e}^{(k)}_{36}) E^{(k)}_z dz \\ (M^P_x, M^P_y, M^P_{xy}) &= \int_{-h/2}^{h/2} (\bar{e}_{31}, \bar{e}_{32}, \bar{e}_{36}) E_z z dz \\ &= \sum_{k=1}^{Na} \int_{z_k}^{z_{k+1}} (\bar{e}^{(k)}_{31}, \bar{e}^{(k)}_{32}, \bar{e}^{(k)}_{36}) E^{(k)}_z z dz \end{aligned} \quad (11)$$

$$\begin{Bmatrix} Q^P_y \\ Q^P_x \end{Bmatrix} = \sum_{k=1}^{Na} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{e}_{14} & \bar{e}_{24} & 0 \\ \bar{e}_{15} & \bar{e}_{25} & 0 \end{bmatrix}^{(k)} \begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}^{(k)} dz$$

where N_a is the number of actuating layers. In Eq. (9a) the matrices [A], [B], and [D] are defined as:

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}$$

$$[B] = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \quad (12)$$

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$$

where

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}^{(k)}_{ij} (1, z, z^2) dz$$

$$i, j = 1, 2, 6 \quad (13)$$

$$A_{ij} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{C}^{(k)}_{ij} dz \quad i, j = 4, 5$$

Here, plate equations of motion are adapted to obtain beam equations of motion. It is more reasonable for a beam to let N_y and M_y be equal to zero. If these assumptions invoked to Eq. (9a), yields:

$$\begin{Bmatrix} N_x \\ N_{xy} \\ M_x \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{16} & \bar{B}_{11} & \bar{B}_{16} \\ \bar{A}_{16} & \bar{A}_{66} & \bar{B}_{16} & \bar{B}_{66} \\ \bar{B}_{11} & \bar{B}_{16} & \bar{D}_{11} & \bar{D}_{16} \\ \bar{B}_{16} & \bar{B}_{66} & \bar{D}_{16} & \bar{D}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^0_x \\ \gamma^0_{xy} \\ \kappa_x \\ \kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} \bar{N}^P_x \\ \bar{N}^P_{xy} \\ \bar{M}^P_x \\ \bar{M}^P_{xy} \end{Bmatrix} \quad (14)$$

where

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{16} & \bar{B}_{11} & \bar{B}_{16} \\ \bar{A}_{16} & \bar{A}_{66} & \bar{B}_{16} & \bar{B}_{66} \\ \bar{B}_{11} & \bar{B}_{16} & \bar{D}_{11} & \bar{D}_{16} \\ \bar{B}_{16} & \bar{B}_{66} & \bar{D}_{116} & \bar{D}_{66} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{16} & B_{11} & B_{16} \\ A_{16} & A_{66} & \bar{B}_{16} & \bar{B}_{66} \\ B_{11} & B_{16} & D_{11} & D_{16} \\ B_{16} & B_{66} & D_{116} & D_{66} \end{bmatrix} - \begin{bmatrix} A_{12} & B_{12} \\ A_{26} & B_{26} \\ B_{12} & D_{12} \\ B_{26} & D_{26} \end{bmatrix} \begin{bmatrix} A_{22} & B_{22} \\ B_{22} & D_{22} \end{bmatrix}^{-1} \begin{bmatrix} A_{12} & B_{12} \\ A_{26} & B_{26} \\ B_{12} & D_{12} \\ B_{26} & D_{26} \end{bmatrix}^T \quad (15)$$

and

$$\begin{Bmatrix} \bar{N}^p_x \\ \bar{N}^p_{xy} \\ \bar{M}^p_x \\ \bar{M}^p_{xy} \end{Bmatrix} = \begin{bmatrix} A_{12} & B_{12} \\ A_{26} & B_{26} \\ B_{12} & D_{12} \\ B_{26} & D_{26} \end{bmatrix} \begin{bmatrix} A_{22} & B_{22} \\ B_{22} & D_{22} \end{bmatrix}^{-1} \begin{Bmatrix} N^p_y \\ M^p_y \end{Bmatrix} - \begin{Bmatrix} N^P_x \\ N^P_{xy} \\ M^P_x \\ M^P_{xy} \end{Bmatrix} \quad (16)$$

Next, in order to find the governing equations of equilibrium Eqs. (9b) and (14) must be substituted into Eqs. (4), with $\partial/\partial y = 0$, to obtain:

$$\bar{A}_{11} u''_0 + \bar{A}_{16} v''_0 + \bar{B}_{11} \psi''_x + \bar{B}_{16} \psi''_y = \frac{d\bar{N}^P_x}{dx}$$

$$\bar{A}_{16} u''_0 + \bar{A}_{66} v''_0 + \bar{B}_{16} \psi''_x + \bar{B}_{66} \psi''_y = \frac{d\bar{N}^P_{xy}}{dx}$$

$$\bar{B}_{11}u_0'' + \bar{B}_{16}v_0'' + \bar{D}_{11}\psi_x'' - k^2 A_{55}\psi_x + \bar{D}_{16}\psi_y''$$
$$- k^2 A_{45}\psi_y - k^2 A_{55} w' = \frac{d\bar{M}_x^P}{dx} - Q_x^P$$
$$\bar{B}_{16}u_0'' + \bar{B}_{66}v_0'' + \bar{D}_{16}\psi_x'' - k^2 A_{45}\psi_x + \bar{D}_{66}\psi_y''$$
$$- k^2 A_{44}\psi_y - k^2 A_{45} w' = \frac{d\bar{M}_{xy}^P}{dx} - Q_y^P$$
$$k^2 A_{55}\psi_x' + k^2 A_{45}\psi_y' + k^2 A_{55} w'' = -q(x) \tag{17}$$

where a prime indicates differentiation with respect to *x*.

2.2. FSDBT2

Here it is assumed that the displacement field of the beam may be represented as:

$$\begin{aligned} u_1(x,y,z) &= u_0(x) + z\psi_x(x) \\ u_2(x,y,z) &= v_0(x) + z\psi_y(x) \\ u_3(x,y,z) &= w(x) \end{aligned} \tag{18}$$

Upon substitution of Eqs. (18) into the linear strain-displacement relations of elasticity, the following results will be obtained:

$$\begin{aligned} &\varepsilon_x = \varepsilon_x^0 + z\kappa_x, \quad \varepsilon_y = \varepsilon_y^0 + z\kappa_y, \quad \varepsilon_z = 0 \\ &\gamma_{yz} = \gamma_{yz}^0, \quad \gamma_{xz} = \gamma_{xz}^0, \quad \gamma_{xy} = \gamma_{xy}^0 + z\kappa_{xy} \end{aligned} \tag{19}$$

where

$$\begin{aligned} &\varepsilon_x^0 = u_0', \quad \kappa_x = \psi_x', \quad \varepsilon_y^0 = 0, \quad \kappa_y = 0 \\ &\gamma_{yz}^0 = \psi_y, \quad \gamma_{xz}^0 = \psi_x + w' \\ &\gamma_{xy}^0 = v_0', \quad \kappa_{xy} = \psi_y' \end{aligned} \tag{20}$$

As far as the stress components are concerned, it is seen from Eqs. (19) and (20) that only σ_z and σ_y are needed to be assumed to vanish. That is:

$$\sigma_z = 0 \tag{21a}$$
$$\sigma_y = 0 \tag{21b}$$

It is to be noted that the first assumption (21a) is also made in the classical, first-order, and third-order shear deformation laminated plate and shell theories. Hence, in the theory developed in the present work Eq. (21b) is the only additional assumption made as far as stresses are concerned.

From Eq. (7a) the normal stress in the *y* direction is:

$$\sigma_y^{(k)} = \bar{Q}_{12}^{(k)}\varepsilon_x^{(k)} + \bar{Q}_{22}^{(k)}\varepsilon_y^{(k)} + \bar{Q}_{26}^{(k)}\gamma_{xy}^{(k)} - \bar{e}_{32}^{(k)}E_z^{(k)} \tag{22}$$

Next, invoking the assumption (21b) in (22) results in:

$$\varepsilon_y^{(k)} = -\frac{1}{\bar{Q}_{22}^{(k)}}(\bar{Q}_{12}^{(k)}\varepsilon_x^{(k)} + \bar{Q}_{26}^{(k)}\gamma_{xy}^{(k)} - \bar{e}_{32}^{(k)}E_z^{(k)}) \tag{23}$$

Now by substituting $\varepsilon_y^{(k)}$ form Eq. (23) into Eq. (7a) we obtain:

$$\begin{aligned} \sigma_x^{(k)} &= \bar{\bar{C}}_{11}^{(k)}\varepsilon_x^{(k)} + \bar{\bar{C}}_{16}^{(k)}\gamma_{xy}^{(k)} - \bar{\bar{e}}_{31}^{(k)}E_z^{(k)} \\ \sigma_{xy}^{(k)} &= \bar{\bar{C}}_{16}^{(k)}\varepsilon_x^{(k)} + \bar{\bar{C}}_{66}^{(k)}\gamma_{xy}^{(k)} - \bar{\bar{e}}_{36}^{(k)}E_z^{(k)} \end{aligned} \tag{24}$$

where

$$\begin{aligned} &\bar{\bar{C}}_{11}^{(k)} = \bar{Q}_{11}^{(k)} - \frac{\bar{Q}_{12}^{(k)2}}{\bar{Q}_{22}^{(k)}}, \quad \bar{\bar{C}}_{16}^{(k)} = \bar{Q}_{16}^{(k)} - \frac{\bar{Q}_{12}^{(k)}\bar{Q}_{26}^{(k)}}{\bar{Q}_{22}^{(k)}} \\ &\bar{\bar{C}}_{66}^{(k)} = \bar{Q}_{66}^{(k)} - \frac{\bar{Q}_{26}^{(k)2}}{\bar{Q}_{22}^{(k)}} \\ &\bar{\bar{e}}_{31}^{(k)} = \bar{e}_{31}^{(k)} - \frac{\bar{Q}_{12}^{(k)}}{\bar{Q}_{22}^{(k)}}\bar{e}_{32}^{(k)}, \quad \bar{\bar{e}}_{36}^{(k)} = \bar{e}_{36}^{(k)} - \frac{\bar{Q}_{26}^{(k)}}{\bar{Q}_{22}^{(k)}}\bar{e}_{32}^{(k)} \end{aligned} \tag{25}$$

Next, using the principle of minimum total potential energy the equilibrium equations can be shown to be:

$$\begin{aligned} &\delta u_0 : \frac{dN_x}{dx} = 0 \\ &\delta v_0 : \frac{dN_{xy}}{dx} = 0 \\ &\delta \psi_x : \frac{dM_x}{dx} - Q_x = 0 \\ &\delta \psi_y : \frac{dM_{xy}}{dx} - Q_y = 0 \\ &\delta w : \frac{dQ_x}{dx} + q(x) = 0 \end{aligned} \tag{26}$$

where $q(x)$ is the applied transverse load at *z*=-*h*/2. The force and moment resultants in Eqs. (26) are defined as Eqs. (6). Now substituting (24) and (7b) (and by using Eq. (19)) into (6) results in:

$$\begin{Bmatrix} N_x \\ N_{xy} \\ M_x \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{\bar{A}}_{11} & \bar{\bar{A}}_{16} & \bar{\bar{B}}_{11} & \bar{\bar{B}}_{16} \\ \bar{\bar{A}}_{16} & \bar{\bar{A}}_{66} & \bar{\bar{B}}_{16} & \bar{\bar{B}}_{66} \\ \bar{\bar{B}}_{11} & \bar{\bar{B}}_{16} & \bar{\bar{D}}_{11} & \bar{\bar{D}}_{16} \\ \bar{\bar{B}}_{16} & \bar{\bar{B}}_{66} & \bar{\bar{D}}_{16} & \bar{\bar{D}}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} \bar{\bar{N}}_x^P \\ \bar{\bar{N}}_{xy}^P \\ \bar{\bar{M}}_x^P \\ \bar{\bar{M}}_{xy}^P \end{Bmatrix} \tag{27a}$$

and

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = k^2 \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} - \begin{Bmatrix} Q_y^p \\ Q_x^p \end{Bmatrix} \tag{27b}$$

where

$$\left(\bar{\bar{A}}_{ij},\bar{\bar{B}}_{ij},\bar{\bar{D}}_{ij}\right)=\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}\bar{\bar{Q}}_{ij}^{(k)}(1,z,z^2)\,dz$$

$$i,j=1,2,6$$

$$\left(\bar{\bar{N}}_x^P,\bar{\bar{N}}_{xy}^P\right)=\sum_{k=1}^{Na}\int_{z_k}^{z_{k+1}}\left(\bar{\bar{e}}_{31}^{(k)},\bar{\bar{e}}_{36}^{(k)}\right)dz \tag{28}$$

$$\left(\bar{\bar{M}}_x^P,\bar{\bar{M}}_{xy}^P\right)=\sum_{k=1}^{Na}\int_{z_k}^{z_{k+1}}\left(\bar{\bar{e}}_{31}^{(k)},\bar{\bar{e}}_{36}^{(k)}\right)z\,dz$$

Finally, the governing equilibrium equations in FSDBT2 are obtained by substituting Eqs. (27) into (26):

$$\bar{\bar{A}}_{11}u_0''+\bar{\bar{A}}_{16}v_0''+\bar{\bar{B}}_{11}\psi_x''+\bar{\bar{B}}_{16}\psi_y''=\frac{d\bar{\bar{N}}_x^P}{dx}$$

$$\bar{\bar{A}}_{16}u_0''+\bar{\bar{A}}_{66}v_0''+\bar{\bar{B}}_{16}\psi_x''+\bar{\bar{B}}_{66}\psi_y''=\frac{d\bar{\bar{N}}_{xy}^P}{dx}$$

$$\bar{\bar{B}}_{11}u_0''+\bar{\bar{B}}_{16}v_0''+\bar{\bar{D}}_{11}\psi_x''-k^2A_{55}\psi_x+\bar{\bar{D}}_{16}\psi_y''$$

$$-k^2A_{45}\psi_y-k^2A_{55}w'=\frac{d\bar{\bar{M}}_x^P}{dx}-Q_x^P \tag{29}$$

$$\bar{\bar{B}}_{16}u_0''+\bar{\bar{B}}_{66}v_0''+\bar{\bar{D}}_{16}\psi_x''-k^2A_{45}\psi_x+\bar{\bar{D}}_{66}\psi_y''$$

$$-k^2A_{44}\psi_y-k^2A_{45}w'=\frac{d\bar{\bar{M}}_{xy}^P}{dx}-Q_y^P$$

$$k^2A_{55}\psi_x'+k^2A_{45}\psi_y'+k^2A_{55}w''=-q(x)$$

By comparing Eqs. (17) with Eqs. (29), it is observed that FSDBT1 and FSDBT2 result in similar equations of equilibrium. Each of these system of equations is five coupled second-order ordinary differential equations that can be solved for any sets of boundary conditions. Solutions of these equations, for the sake of brevity, will not be taken up here.

In electrical loading case, we suppose that the electric field vary linearly within the *k*th actuator layer (see Fig. 2). That is;

$$E_z^{(k)}=E_1^k(x)\psi_1^k(z)+E_2^k(x)\psi_2^k(z) \tag{30}$$

where the linear interpolation functions of the *k*th layer (i.e. ψ_i^k) are defined as:

$$\psi_1^k=\frac{z_{k+1}-z}{h_k},\quad \psi_2^k=\frac{z-z_k}{h_k},\quad z_k\le z\le z_{k+1} \tag{31}$$

and E_1^k and E_2^k denote the electric field at $z=z_k$ and $z=z_{k+1}$ of the *k*th actuator layer (see Fig. 2). In Eqs. (31) h_k is the thickness of the *k*th actuator layer.

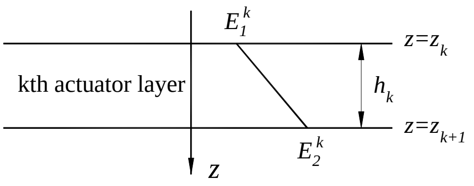

Figure 2. Linear distribution of electric field intensity through the thickness of an actuator layer

Next, the electric forces and moments can be evaluated as:

$$\begin{Bmatrix}N_x^P\\N_y^P\\N_{xy}^P\end{Bmatrix}=\frac{1}{2}\sum_{k=1}^{Na}\begin{bmatrix}\bar{Q}_{11}&\bar{Q}_{12}&\bar{Q}_{16}\\\bar{Q}_{12}&\bar{Q}_{22}&\bar{Q}_{26}\\\bar{Q}_{16}&\bar{Q}_{26}&\bar{Q}_{66}\end{bmatrix}^{(k)}\begin{Bmatrix}\bar{d}_{31}\\\bar{d}_{32}\\\bar{d}_{36}\end{Bmatrix}^{(k)}. \tag{32a}$$

$$(E_1^k+E_2^k)$$

$$\begin{Bmatrix}N_x^P\\N_y^P\\N_{xy}^P\end{Bmatrix}=\frac{1}{6}\sum_{k=1}^{Na}\begin{bmatrix}\bar{Q}_{11}&\bar{Q}_{12}&\bar{Q}_{16}\\\bar{Q}_{12}&\bar{Q}_{22}&\bar{Q}_{26}\\\bar{Q}_{16}&\bar{Q}_{26}&\bar{Q}_{66}\end{bmatrix}^{(k)}\begin{Bmatrix}\bar{d}_{31}\\\bar{d}_{32}\\\bar{d}_{36}\end{Bmatrix}^{(k)}. \tag{32b}$$

$$\left[E_1^k(h_k+3z_k)+E_2^k(2h_k+3z_k)\right]h_k$$

If electric field intensity through the thickness of each layer be constant, then it can be approximated by:

$$E_1^k=E_2^k=\frac{V_k}{h_k} \tag{33}$$

3. NUMERICAL RESULTS

The effectiveness of the new beam theory FSDBT2 is demonstrated through examples of static bending of general laminated composite beams subjected to electro-mechanical loadings. The assessment of accuracy of FSDBT2 for the case of bending of laminated beams will be obtained by comparing the results with those obtained by FSDBT1.

Several numerical examples are solved for laminated composite beams consist of piezoelectric layers bonded on the top and bottom surfaces of the beam. Graphite/epoxy composite material and PZT-4 are selected for the substrate orthotropic layers and piezoelectric layers, respectively. The material properties for graphite/epoxy T300/5208 orthotropic layers of the substrate are [15]:

$$E_1=132\,\text{GPa}\,,\quad E_2=10.8\,\text{GPa}$$

$$G_{12} = 5.65\,\text{GPa}\,,\quad G_{23} = 3.38\,\text{GPa}$$
$$\nu_{12} = 0.24\,,\quad \nu_{23} = 0.59 \tag{34}$$

where the subscripts 1, 2, and 3 indicate the on-axis (i.e., principal) material coordinates. Also material properties for PZT-4 piezoelectric layers are [16]:

$$[C] = \begin{bmatrix} 139 & 77.8 & 74.3 & 0 & 0 & 0 \\ 78 & 139 & 74.3 & 0 & 0 & 0 \\ 74.3 & 74.3 & 115 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 25.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 30.6 \end{bmatrix} \text{GPa}$$
$$[e] = \begin{bmatrix} 0 & 0 & 0 & 0 & 12.7 & 0 \\ 0 & 0 & 0 & 12.7 & 0 & 0 \\ -5.2 & -5.2 & 15.1 & 0 & 0 & 0 \end{bmatrix} \text{C/m}^2 \tag{35}$$

The laminated beams are assumed to have the total thickness h=0.06m and length L=10h. All orthotropic layers and also piezoelectric layers are assumed to have equal thicknesses. For all numerical examples, it is assumed that the beams have simply supported boundary conditions. In electrical loading case, the same voltage is applied across the two actuator layers with opposite polarity.

Figs. 3 and 4 show through the thickness distributions of normalized axial stress σ_x and transverse shear stress σ_{xz}, respectively, at $x=L/3$ of a $[P/0^o/90^o]_s$ laminated composite beam under uniform transverse load. Also through the thickness distributions of normalized axial stress σ_x and transverse shear stress σ_{xz} at $x=L/3$ of a $[P/0^o/90^o]_s$ laminated composite beam under electrical loading are shown in Figs. 5 and 6, respectively. It is noted that P in $[P/0^o/90^o]_s$ denotes piezoelectric layer with 0^o rotation with respect to the x axis. It is observed that in electrical loading the transverse shear stress σ_{xz} is equal to zero.

The numerical results of Figs. 3-6 indicate that FSDBT1 and FSDBT2 give us completely similar results for stress components of a cross ply laminated composite beam subjected to both mechanical and electrical loadings.

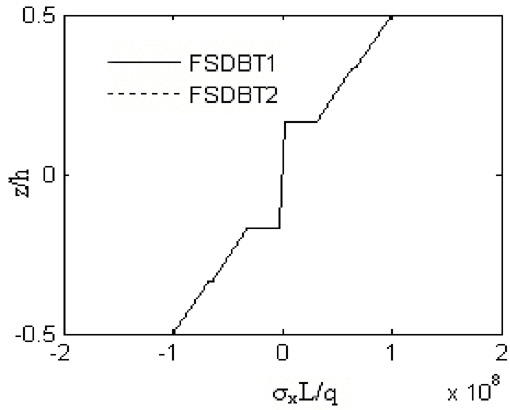

Figure 3. Through the thickness distribution of σ_x at $x=L/3$ of a $[P/0^o/90^o]_s$ laminated beam under uniform transverse loading

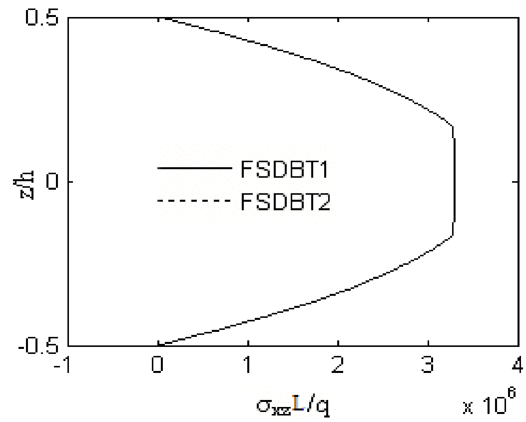

Figure 4. Through the thickness distribution of σ_{xz} at $x=L/3$ of a $[P/0^o/90^o]_s$ laminated beam under uniform transverse load

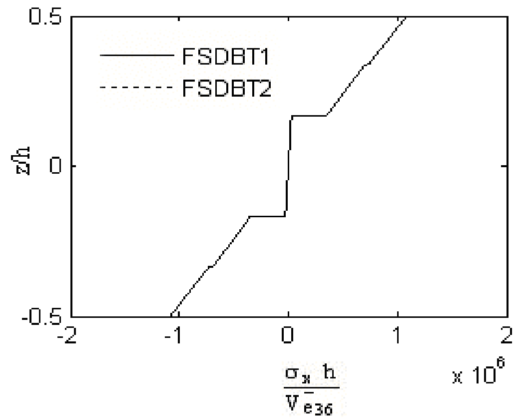

Figure 5. Through the thickness distribution of σ_x at $x=L/3$ of a $[P/0^o/90^o]_s$ laminated beam under electrical loading

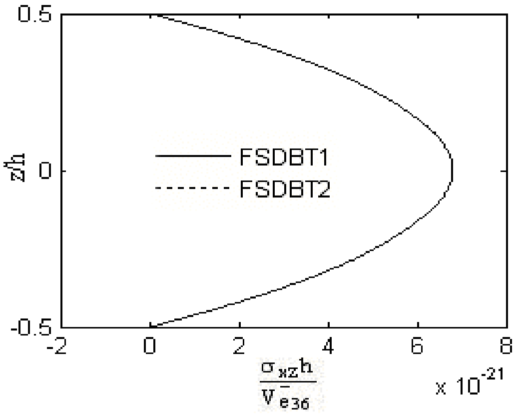

Figure 6. Through the thickness distribution of σ_{xz} at $x=L/3$ of a $[P/0^o/90^o]_s$ laminated beam under electrical loading

Next, in order to verify the correctness and accuracy of FSDBT2 a laminated beam with a general lamination is considered. Here, for example, we consider a $[P/20^o/40^o/60^o/80^o/P]$ laminated beam subjected to electro-mechanical loadings. The variations of normalized axial stress σ_x and transverse shear stress σ_{xz} at $x=L/3$ through the thickness of a beam with general lamination $[P/20^o/40^o/60^o/80^o/P]$ subjected to a uniform transverse load are shown in Figs. 7 and 8, respectively. Similar result for normalized axial stress σ_x of the aforementioned laminated beam subjected to electrical loading is displayed in Fig. 9. It is seen from Figs. 7-9 that there are excellent agreements between the FSDBT1 and FSDBT2 for this unsymmetric laminated composite beam with piezoelectric layers for both mechanical and electrical loadings.

Finally, Fig. 10 presents the distributions of normalized transverse shear stress σ_{xz} through the thickness of $[P/20^o/40^o/60^o/80^o/P]$ laminated beam subjected to a uniform transverse load at various values of the length coordinate *x*.

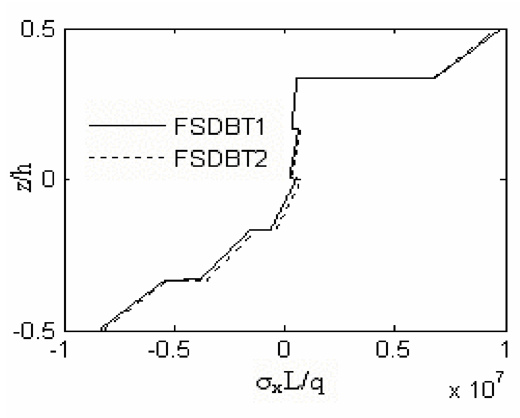

Figure 7. Through the thickness distribution of σ_x at $x=L/3$ of a $[P/20^o/40^o/60^o/80^o/P]$ laminated beam under uniform transverse loading

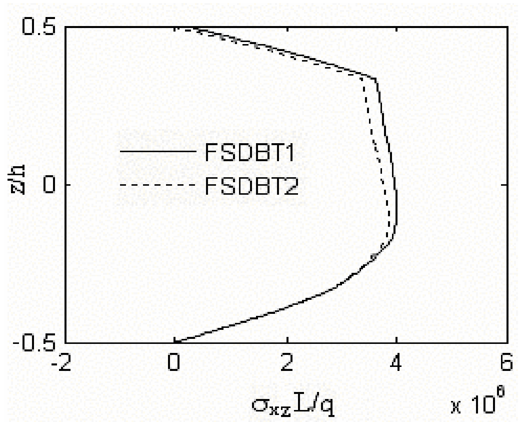

Figure 8. Through the thickness distribution of σ_{xz} at $x=L/3$ of a $[P/20^o/40^o/60^o/80^o/P]$ laminated beam under uniform transverse loading

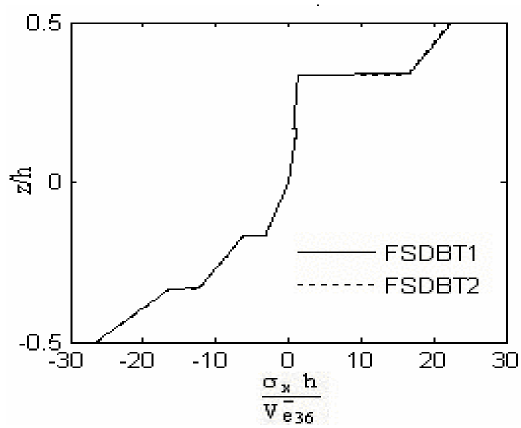

Figure 9. Through the thickness distribution of σ_x at $x=L/3$ of a $[P/20^o/40^o/60^o/80^o/P]$ laminated beam under electrical loading

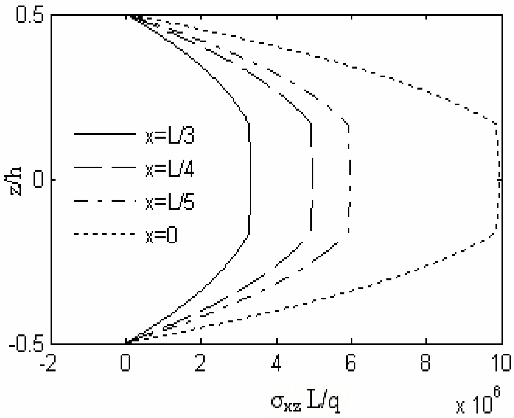

Figure 10. Through the thickness distribution of σ_{xz} at various values of the length coordinate *x* of a [*P*/20°/40°/60°/80°/*P*] laminated beam under uniform transverse loading

4. CONCLUSIONS

Within a first-order shear deformation theory, a new laminated beam theory with general lamination is developed. The structure consists of piezoelectric layers bonded on the top and bottom surfaces of the laminated composite beam. The approach adopted in the derivation of the equilibrium equations in the new beam theory is direct and straightforward similar to the ones used in developing laminated plate and shell theories. The ideas developed in the present work may readily be used in developing higher-order shear deformation and layerwise laminated beam theories. For the assessment of the accuracy of this theory, analytical solutions are obtained and compared with those of the existing first-order beam theory. It is found that the new beam theory can predict accurately displacements and stresses in the beams subjected to both mechanical and electrical loadings.

REFERENCES

1. Abramovich, H., Livshits, A., 1993, “Dynamic Behavior of Cross-Ply Laminated Beams with Piezoelectric Layers”, *Composite Structures*, 25, 371–379.
2. Abramovich, H., 1998, “Deflection Control of Laminated Composite Beams with Piezoceramic Layers – Closed Form Solutions”, *Composite Structures*, 43, 217–231.
3. Rammerstorfer, F.G., 1977, “Increase of the First Natural Frequency and Buckling Load of Plates by Optimal Fields of Initial Stresses”, *Acta Mechanica*, 27, 217–38.
4. Almeida, S.F.M., 1999, “Shape Control of Laminated Plates with Piezoelectric Actuators Including Stress-Stiffening Effects”, *AIAA J*ournal, 37(8), 1017–1019.
5. Waisman, H., Abramovich, H., 2002, “Active Stiffening of Laminated Composite Beams using Piezoelectric Actuators”, *Composite Structures*, 58, 109–120.
6. Gaudenzi, P., Carbonaro, R., Benzi, E., 2000, “Control of Beam Vibrations by means of Piezoelectric Devices: Theory and Experiments,” *Composite Structures*, 50, 373-379.
7. Huang, D., Sun, B.H., 2001, “Approximate Solution on Smart Composite Beams by using MATLAB”, *Composite Structures*, 54, 197-205.
8. Tong, L., Luo, Q., 2003, “Exact Dynamic Solutions to Piezoelectric Smart Beams Including Peel Stresses Part I: Theory and Application”, *International Journal of Solids and Structures*, 40, 4789–4812.
9. Luo, Q., Tong, L., 2003, “Exact Dynamic Solutions to Piezoelectric Smart Beams Including Peel Stresses Part II: Numerical Results, Comparison and Approximate Solution”, *International Journal of Solids and Structures*, 40, 4813–4836.
10. Vel, S.S., Mewer,, R.C., Batra, R.C., 2004, Analytical Solution for the Cylindrical Bending Vibration of Piezoelectric Composite Plates”, *International Journal of Solids and Structures*, 41, 1625–1643.
11. Pan, E., Heyliger, P.R., 2003, “Exact Solutions for Magneto-Electro-Elastic Laminates in Cylindrical Bending”, *International Journal of Solids and Structures*, 40, 6859–6876.
12. Aldraihem, O.J., Khdeir, A.A., 2003, “Exact Deflection Solutions of Beams with Shear Piezoelectric Actuators”, *International Journal of Solids and Structures*, 40, 1–12.
13. Tahani, M., In Press, “Analysis of Laminated Composite Beams using Layerwise Displacement Theories”, *Composite Structures*.
14. Reddy, J.N., 1999, On Laminated Composite Plates with Integrated Sensors and Actuators”, *Engineering Structures*, 21, 568-593.
15. Herakovich, C.T., 1998, *Mechanics of Fibrous Composites*, John Wiley and Sons, Inc., New York.
16. Cheng, Z.Q., Lim, C.W., Kitipornchai, S., 2000, “Three-Dimensional Asymptotic Approach to Inhomogeneous and Laminated Piezoelectric Plates”, *International Journal of Solids and Structures*, 37, 3153–3175.