ANALYTICAL STUDY OF PIEZOELECTRIC ACTUATED LAMINATES UNDER TRANSVERSE LOADING

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ABSTRACT

This paper deals with analytical solution of piezoelectric laminated composite plates with arbitrary lamination and boundary conditions subjected to electromechanical loadings. A new first-order shear deformation plate theory is developed based on separation of spatial variables of displacement field components. Two systems of coupled ordinary differential equations with constant coefficients are obtained by using the principle of minimum total potential energy. The obtained equations are solved analytically with the aid of the state-space approach. The results obtained from this theory are compared with the Levy-type solutions of antisymmetric angle-ply laminates with various admissible boundary conditions to verify the validity and accuracy of the present theory. It is seen that the present results have excellent agreements with those obtained by Levy-type method. Since the procedure used is simple and straightforward, it can readily be adopted in developing higher-order shear deformation and layerwise laminated plate and shell theories.

1. INTRODUCTION

Since about two decades ago, several papers have been devoted to the theoretical study of piezoelectric laminated plates and beams. Crawley and de Luis [1] studied the electroelastical properties of a piezoelectric/elastic laminated beam with its extension mechanism. Lee and Moon [2] and Lee [3] used Kirchhoff plate theory to develop a simple theory for laminated plates with distributed piezoelectric layers. Wang and Rogers [4] developed analytical solutions based on the classical laminate plate theory (CLPT) for plates with surface-bonded or embedded piezoelectric layers. Jonnalagadda et al. [5] employed first-order shear deformation theory (FSDT) to solve the piezothermoelastic response of hybrid plates. Mitchell and Reddy [6] used higher-order shear deformation theory (HSDT) based on single-layer theory for mechanical displacement of rectangular hybrid laminates. A coupled, FSDT for multilayered piezoelectric plates was proposed by Huang and Wu [7]. Kapuria et al. [8] presented a Levy-type solution for the bending of cross-ply, hybrid, plates using a mixed formulation of FSDT and CLPT. Zhang and Sun [9] utilized the variation principle to derive the governing equations of sandwich plates containing a piezoelectric core using the shear mode of piezoelectric materials. Vel and Batra [10-12] provided the analytical solutions to study the generalized plane strain deformations of piezoelectric laminated plates subjected to arbitrary boundary conditions via Eshelby–Stroh formalism. By the use of the trigonometric series, Zhang et al. [13] obtained exact solutions of thermoelectroelastic laminates with simply supported boundaries. Recently, Cheng et al. [14] obtained governing equations for the anisotropic piezoelectric laminated plate, based on Hamilton's principle and assumption of Reddy's simple high-order theory.

The purpose of the present study is to develop an analytical method for bending analysis of piezoelectric laminated plates with arbitrary lamination and boundary conditions. As the numerical result, an anisymmetric angle-ply hybrid plate under various sets of boundary conditions is examined. The comparison of the results with those obtained from the Levy-type solution shows an excellent agreement. The approach adopted in the present work will be demonstrated within the framework of a FSDT. However, the idea is straightforward and general and can readily be used in developing higher-order shear deformation and layerwise laminated plate and shell theories.

2. FORMULATION OF THE PROBLEM

2.1- Strain field

Figure 1 shows an undeformed plate of uniform thickness h in a Cartesian coordinate system (x,y,z), where the midplane of the plate coincides with z = 0. The plate has a width b in the lateral (y-) direction, and length a in the longitudinal (x-) direction. It is composed of arbitrary N orthotropic layers which some of them can be piezoelectric. Here, the theory will be developed within the framework of the FSDT [15]. To this end, it is assumed that the components of the displacement field of the plate may be presented as:

$$\begin{aligned} u(x,y,z) &= u_i(x)\bar{u}_i(y) + z\psi_i(x)\bar{\psi}_i(y) \\ v(x,y,z) &= v_i(x)\bar{v}_i(y) + z\phi_i(x)\bar{\phi}_i(y) \qquad i = 1,2,...,n \\ w(x,y) &= w_i(x)\bar{w}_i(y) \end{aligned} \tag{1}$$

where, for the sake of brevity, the Einstein summation convention has been introduced – a repeated index indicates summation over all values of that index. In equations (1) $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ are, respectively, the displacements in x, y, and z directions, and $u_i(x)$, $\bar{u}_i(y)$, $v_i(x)$, $\bar{v}_i(y)$, $\psi_i(x)$, $\bar{\psi}_i(y)$, $\phi_i(x)$, $\bar{\phi}_i(y)$, $w_i(x)$, and $\bar{w}_i(y)$ are unknown functions. Also n is the total number of terms considered in the summation.

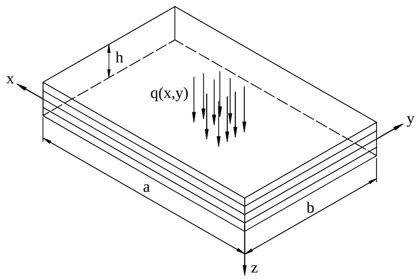

Figure 1 : The plate geometry and coordinate system.

Substitution of the displacement field (1) into the linear strain-displacement relations of elasticity yields a new form of strain-displacement relations as below:

$$\begin{aligned} &\varepsilon_x = u_i\bar{u}_i' + z\,\psi_i\bar{\psi}_i' = \varepsilon_x^0 + z\,\kappa_x\,, \qquad \varepsilon_y = v_i\bar{v}_i' + z\,\phi_i\bar{\phi}_i' = \varepsilon_y^0 + z\,\kappa_y \\ &\gamma_{yz} = \phi_i\bar{\phi}_i + w_i\bar{w}_i' = \gamma_{yz}^0\,, \qquad \gamma_{xz} = \psi_i\bar{\psi}_i + w_i'\bar{w}_i = \gamma_{xz}^0 \qquad\qquad i=1,2,\ldots,n \\ &\gamma_{xy} = u_i\bar{u}_i' + v_i'\bar{v}_i + z(\psi_i\bar{\psi}_i' + \phi_i'\bar{\phi}_i) = \gamma_{xy}^0 + z\kappa_{xy}\,, \qquad \varepsilon_z = 0 \end{aligned} \tag{2}$$

2.2- Equilibrium equations

Next, using the principle of minimum total potential energy [16], equilibrium equations and boundary conditions corresponding to the independent variables can be shown to be:

$$\begin{aligned} &\delta u_i: \ \frac{dN_x^i}{dx} - N_{xy1}^i = 0\,, \qquad \delta v_i: \ \frac{dN_{xy2}^i}{dx} - N_y^i = 0 \\ &\delta\psi_i: \ \frac{dM_x^i}{dx} - M_{xy1}^i - Q_{x1}^i = 0\,, \qquad \delta\phi_i: \ \frac{dM_{xy2}^i}{dx} - M_y^i - Q_{y1}^i = 0 \\ &\delta w_i: \ \frac{dQ_{x2}^i}{dx} - Q_{y2}^i + q_i(x) = 0 \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\delta\bar{u}_i: \ \frac{d\bar{N}_{xy1}^i}{dy} - \bar{N}_x^i = 0\,, \qquad \delta\bar{v}_i: \ \frac{d\bar{N}_y^i}{dy} - \bar{N}_{xy2}^i = 0 \\ &\delta\bar{\psi}_i: \ \frac{d\bar{M}_{xy1}^i}{dy} - \bar{M}_x^i - \bar{Q}_{x1}^i = 0\,, \qquad \delta\bar{\phi}_i: \ \frac{d\bar{M}_y^i}{dy} - \bar{M}_{xy2}^i - \bar{Q}_{y1}^i = 0 \end{aligned}$$

$$\delta\overline{w}_i: \quad \frac{d\overline{Q}^i_{y2}}{dy} - \overline{Q}^i_{x2} + \overline{q}_i(y) = 0 \tag{4}$$

In the above equations the generalized stress resultants, $q_i(x)$, and $\overline{q}_i(y)$ are defined as:

$$\begin{bmatrix} \{N^i\}^T \\ \{M^i\}^T \\ \{\Theta^i\}^T \end{bmatrix} = \begin{bmatrix} N^i_x & N^i_y & N^i_{xy1} & N^i_{xy2} \\ M^i_x & M^i_y & M^i_{xy1} & M^i_{xy2} \\ Q^i_{y1} & Q^i_{y2} & Q^i_{x1} & Q^i_{x2} \end{bmatrix} = \int_0^b \begin{bmatrix} N_x\overline{u}_i & N_y\overline{v}'_i & N_{xy}\overline{u}'_i & N_{xy}\overline{v}_i \\ M_x\overline{\psi}_i & M_y\overline{\phi}'_i & M_{xy}\overline{\psi}'_i & M_{xy}\overline{\phi}_i \\ Q_y\overline{\phi}_i & Q_y\overline{w}'_i & Q_x\overline{\psi}_i & Q_x\overline{w}_i \end{bmatrix} dy \tag{5}$$

$$\begin{bmatrix} \{\overline{N}^i\}^T \\ \{\overline{M}^i\}^T \\ \{\overline{\Theta}^i\}^T \end{bmatrix} = \begin{bmatrix} \overline{N}^i_x & \overline{N}^i_y & \overline{N}^i_{xy1} & \overline{N}^i_{xy2} \\ \overline{M}^i_x & \overline{M}^i_y & \overline{M}^i_{xy1} & \overline{M}^i_{xy2} \\ \overline{Q}^i_{y1} & \overline{Q}^i_{y2} & \overline{Q}^i_{x1} & \overline{Q}^i_{x2} \end{bmatrix} = \int_0^a \begin{bmatrix} N_x u'_i & N_y v_i & N_{xy}u_i & N_{xy}v'_i \\ M_x\psi'_i & M_y\phi_i & M_{xy}\psi_i & M_{xy}\phi'_i \\ Q_y\phi_i & Q_y w_i & Q_x\psi_i & Q_x w'_i \end{bmatrix} dx \tag{6}$$

$$q_i(x) = \int_0^b q(x,y)\overline{w}_i\, dy \;, \quad \overline{q}_i(y) = \int_0^a q(x,y) w_i\, dy \tag{7}$$

In addition, the stress resultants are:

$$(N_x, N_y, N_{xy}, Q_y, Q_x) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yz}, \sigma_{xz}) dz, \quad (M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}) z dz \tag{8}$$

The primary and secondary variables of the theory are:
For edges parallel to y-axis (i.e., x=0,a);

$$\begin{array}{lll} \text{Primary variables:} & u_i, v_i, \psi_i, \phi_i, w_i & \\ \text{Secondary variables:} & N^i_x, N^i_{xy2}, M^i_x, M^i_{xy2}, Q^i_{x2} & i=1,2,\ldots,n \end{array} \tag{9}$$

For edges parallel to x-axis (i.e., y=0,b);

$$\begin{array}{lll} \text{Primary variables:} & \overline{u}_i, \overline{v}_i, \overline{\psi}_i, \overline{\phi}_i, \overline{w}_i & \\ \text{Secondary variables:} & \overline{N}^i_{xy1}, \overline{N}^i_y, \overline{M}^i_{xy1}, \overline{M}^i_y, \overline{Q}^i_{y2} & i=1,2,\ldots,n \end{array} \tag{10}$$

2.3- Laminate constitutive relations

The linear constitutive relations for the kth lamina of the hybrid laminate are given by [15]:

$$\{\sigma\}^{(k)} = [\overline{Q}]^{(k)}\{\varepsilon\}^{(k)} - [\overline{e}]^{(k)}\{E\}^{(k)} \tag{11}$$

As a large electric potential difference is applied across one or more layers of the laminate, it is assumed that the electric field owing to the variation in stress is insignificant compared with the applied electric field. In equation (11) the electric field vector $\{E\}$ is related to electric potential Φ by:

$$E_j = \Phi_{,j} \tag{12}$$

Also $[\overline{Q}]^{(k)}$ and $[\overline{e}]^{(k)}$ denote the transformed reduced plane-stress stiffness matrix and the matrix of transformed piezoelectric moduli of the kth lamina, respectively. Upon substitution of equations (2) into equation (11) and the subsequent results into equations (8), the stress resultants are obtained which can be presented as follows:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ & & A_{66} & B_{16} & B_{26} & B_{66} \\ & & & D_{11} & D_{12} & D_{16} \\ & sym. & & & D_{22} & D_{26} \\ & & & & & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^o_x \\ \varepsilon^o_y \\ \gamma^o_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} - \begin{Bmatrix} N^P_x \\ N^P_y \\ N^P_{xy} \\ M^P_x \\ M^P_y \\ M^P_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = k^2 \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma^0_{yz} \\ \gamma^0_{xz} \end{Bmatrix} - \begin{Bmatrix} Q^P_y \\ Q^P_x \end{Bmatrix} \tag{13}$$

Here, A_{ij}, B_{ij}, and D_{ij} denote the extensional stiffnesses, the bending-extensional coupling stiffnesses, and the bending stiffnesses, respectively [15]. Also k^2 (=5/6) is the shear correction factor of FSDT and $\{N^P\}$, $\{M^P\}$ and $\{Q^P\}$ are the electric stress resultants:

$$\{N^P\}^T=\sum_{k=1}^{N_a}\int_{z_k}^{z_{k+1}}[\bar{e}_{31}^{(k)},\bar{e}_{32}^{(k)},\bar{e}_{36}^{(k)}]E_z^{(k)}dz\,,\quad \{M^P\}^T=\sum_{k=1}^{N_a}\int_{z_k}^{z_{k+1}}[\bar{e}_{31}^{(k)},\bar{e}_{32}^{(k)},\bar{e}_{36}^{(k)}]E_z^{(k)}z\,dz$$
$$\{Q^P\}=\sum_{k=1}^{N_a}\int_{z_k}^{z_{k+1}}\begin{bmatrix}\bar{e}_{14} & \bar{e}_{24} & 0\\ \bar{e}_{15} & \bar{e}_{25} & 0\end{bmatrix}^{(k)}\{E\}^{(k)}dz \tag{14}$$

Upon substitution of equations (2) into (13) and the subsequent results into equations (5) and (6), the generalized stress resultants are obtained which can be represented as follows:

$$\begin{Bmatrix}\{N^i\}\\ \{M^i\}\end{Bmatrix}=\left[A^{ij}\right]\left\{\xi_j\right\}-\begin{Bmatrix}\{N^{Pi}\}\\ \{M^{Pi}\}\end{Bmatrix},\quad \left\{\Theta^i\right\}=\left[B^{ij}\right]\left\{\eta_j\right\}-\left\{\Theta^{Pi}\right\} \tag{15}$$

$$\begin{Bmatrix}\{\bar{N}^i\}\\ \{\bar{M}^i\}\end{Bmatrix}=\left[\bar{A}^{ij}\right]\left\{\bar{\xi}_j\right\}-\begin{Bmatrix}\{\bar{N}^{Pi}\}\\ \{\bar{M}^{Pi}\}\end{Bmatrix},\quad \left\{\bar{\Theta}^i\right\}=\left[\bar{B}^{ij}\right]\left\{\bar{\eta}_j\right\}-\left\{\bar{\Theta}^{Pi}\right\} \tag{16}$$

where

$$\left\{\xi_j\right\}=\begin{bmatrix}u'_j & v_j & u_j & v'_j & \psi'_j & \phi_j & \psi_j & \phi'_j\end{bmatrix}^T,\quad \left\{\eta_j\right\}=\begin{bmatrix}\phi_j & w_j & \psi_j & w'_j\end{bmatrix}^T \tag{17}$$

$$\left\{\bar{\xi}_j\right\}=\begin{bmatrix}\bar{u}_j & \bar{v}'_j & \bar{u}'_j & \bar{v}_j & \bar{\psi}_j & \bar{\phi}'_j & \bar{\psi}'_j & \bar{\phi}_j\end{bmatrix}^T,\quad \left\{\bar{\eta}_j\right\}=\begin{bmatrix}\bar{\phi}_j & \bar{w}'_j & \bar{\psi}_j & \bar{w}_j\end{bmatrix}^T \tag{18}$$

and the stiffness coefficients A_{mn}^{ij}, B_{mn}^{ij}, $\bar{A}_{mn}^{ij}$, and $\bar{B}_{mn}^{ij}$ are defined by:

$$\left[A^{ij}\right]=\int_0^b\left(\left[\alpha\right]\otimes\left\{\bar{\xi}_i\right\}\left\{\bar{\xi}_j\right\}^T\right)dy\,,\quad \left[B^{ij}\right]=\int_0^b\left(\left[\beta\right]\otimes\left\{\bar{\eta}_i\right\}\left\{\bar{\eta}_j\right\}^T\right)dy \tag{19}$$

$$\left[\bar{A}^{ij}\right]=\int_0^a\left(\left[\alpha\right]\otimes\left\{\xi_i\right\}\left\{\xi_j\right\}^T\right)dx\,,\quad \left[\bar{B}^{ij}\right]=\int_0^a\left(\left[\beta\right]\otimes\left\{\eta_i\right\}\left\{\eta_j\right\}^T\right)dx \tag{20}$$

where $[\alpha]$ and $[\beta]$ are:

$$[\alpha]=\begin{bmatrix}A_{11} & A_{12} & A_{16} & A_{16} & B_{11} & B_{12} & B_{16} & B_{16}\\ & A_{22} & A_{26} & A_{26} & B_{12} & B_{22} & B_{26} & B_{26}\\ & & A_{66} & A_{66} & B_{16} & B_{26} & B_{66} & B_{66}\\ & & & A_{66} & B_{16} & B_{26} & B_{66} & B_{66}\\ & & & & D_{11} & D_{12} & D_{16} & D_{16}\\ & & sym. & & & D_{22} & D_{26} & D_{26}\\ & & & & & & D_{66} & D_{66}\\ & & & & & & & D_{66}\end{bmatrix},\quad [\beta]=\begin{bmatrix}A_{44} & A_{44} & A_{45} & A_{45}\\ & A_{44} & A_{45} & A_{45}\\ & & A_{55} & A_{55}\\ sym. & & & A_{55}\end{bmatrix} \tag{21}$$

It must be noted that the sign $\otimes$ used in equations (18) and (19) is referred to *array multiplication* of two matrices.

2.4- Governing equations of equilibrium

The equilibrium equations (3) and (4) can be expressed in terms of displacements by substituting the generalized stress resultants from (15) and (16). Hence, two sets of ordinary differential equations will be obtained as follows:

$$\begin{aligned}\delta u_i:\ & A_{11}^{ij}u''_j+(A_{13}^{ij}-A_{31}^{ij})u'_j-A_{33}^{ij}u_j+A_{14}^{ij}v''_j+(A_{12}^{ij}-A_{34}^{ij})v'_j-A_{32}^{ij}v_j+A_{15}^{ij}\psi''_j+(A_{17}^{ij}-A_{35}^{ij})\psi'_j-A_{37}^{ij}\psi_j+A_{18}^{ij}\phi''_j\\ & +(A_{16}^{ij}-A_{38}^{ij})\phi'_j-A_{36}^{ij}\phi_j=\frac{dN_x^{Pi}}{dx}-N_{xy1}^{Pi}\end{aligned}$$

$$\begin{aligned}
&\delta v_i : A_{41}^{ij} u_j'' + (A_{43}^{ij} - A_{21}^{ij}) u_j' - A_{23}^{ij} u_j + A_{44}^{ij} v_j'' + (A_{42}^{ij} - A_{24}^{ij}) v_j' - A_{22}^{ij} v_j + A_{45}^{ij} \psi_j'' + (A_{47}^{ij} - A_{25}^{ij}) \psi_j' - A_{27}^{ij} \psi_j \\
&\qquad + A_{48}^{ij} \phi_j'' + (A_{46}^{ij} - A_{28}^{ij}) \phi_j' - A_{26}^{ij} \phi_j = \frac{dN_{xy2}^{Pi}}{dx} - N_y^{Pi} \\
&\delta \psi_i : A_{51}^{ij} u_j'' + (A_{53}^{ij} - A_{71}^{ij}) u_j' - A_{73}^{ij} u_j + A_{54}^{ij} v_j'' + (A_{52}^{ij} - A_{74}^{ij}) v_j' - A_{72}^{ij} v_j + A_{55}^{ij} \psi_j'' + (A_{57}^{ij} - A_{75}^{ij}) \psi_j' - (A_{77}^{ij} + B_{33}^{ij}) \psi_j \\
&\qquad + A_{58}^{ij} \phi_j'' + (A_{56}^{ij} - A_{78}^{ij}) \phi_j' - (A_{76}^{ij} + B_{31}^{ij}) \phi_j - B_{34}^{ij} w_j' - B_{32}^{ij} w_j = \frac{dM_x^{Pi}}{dx} - M_{xy1}^{Pi} - Q_{x1}^{Pi} \\
&\delta \phi_i : A_{81}^{ij} u_j'' + (A_{83}^{ij} - A_{61}^{ij}) u_j' - A_{63}^{ij} u_j + A_{84}^{ij} v_j'' + (A_{82}^{ij} - A_{64}^{ij}) v_j' - A_{62}^{ij} v_j + A_{85}^{ij} \psi_j'' + (A_{87}^{ij} - A_{65}^{ij}) \psi_j' - (A_{67}^{ij} + B_{13}^{ij}) \psi_j \\
&\qquad + A_{88}^{ij} \phi_j'' + (A_{86}^{ij} - A_{68}^{ij}) \phi_j' - (A_{66}^{ij} + B_{11}^{ij}) \phi_j - B_{14}^{ij} w_j' - B_{12}^{ij} w_j = \frac{dM_{xy2}^{Pi}}{dx} - N_y^{Pi} - Q_{y1}^{Pi} \\
&\delta w_i : B_{43}^{ij} \psi_j' - B_{23}^{ij} \psi_j + B_{41}^{ij} \phi_j' - B_{21}^{ij} \phi_j + B_{44}^{ij} w_j'' + (B_{42}^{ij} - B_{24}^{ij}) w_j' - B_{22}^{ij} w_j = \frac{dQ_{x2}^{Pi}}{dx} - Q_{y2}^{Pi} - q_i(x)
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
&\delta \bar{u}_i : \bar{A}_{33}^{ij} \bar{u}_j'' + (\bar{A}_{31}^{ij} - \bar{A}_{13}^{ij}) \bar{u}_j' - \bar{A}_{11}^{ij} \bar{u}_j + \bar{A}_{32}^{ij} \bar{v}_j'' + (\bar{A}_{34}^{ij} - \bar{A}_{12}^{ij}) \bar{v}_j' - \bar{A}_{14}^{ij} \bar{v}_j + \bar{A}_{37}^{ij} \bar{\psi}_j'' + (\bar{A}_{35}^{ij} - \bar{A}_{17}^{ij}) \bar{\psi}_j' - \bar{A}_{15}^{ij} \bar{\psi}_j + \bar{A}_{36}^{ij} \bar{\phi}_j'' \\
&\qquad + (\bar{A}_{38}^{ij} - \bar{A}_{16}^{ij}) \bar{\phi}_j' - \bar{A}_{18}^{ij} \bar{\phi}_j = \frac{d\bar{N}_{xy1}^{Pi}}{dy} - \bar{N}_x^{Pi} \\
&\delta \bar{v}_i : \bar{A}_{23}^{ij} \bar{u}_j'' + (\bar{A}_{21}^{ij} - \bar{A}_{43}^{ij}) \bar{u}_j' - \bar{A}_{41}^{ij} \bar{u}_j + \bar{A}_{22}^{ij} \bar{v}_j'' + (\bar{A}_{24}^{ij} - \bar{A}_{42}^{ij}) \bar{v}_j' - \bar{A}_{44}^{ij} \bar{v}_j + \bar{A}_{27}^{ij} \bar{\psi}_j'' + (\bar{A}_{25}^{ij} - \bar{A}_{47}^{ij}) \bar{\psi}_j' - \bar{A}_{45}^{ij} \bar{\psi}_j + \bar{A}_{26}^{ij} \bar{\phi}_j'' \\
&\qquad + (\bar{A}_{28}^{ij} - \bar{A}_{46}^{ij}) \bar{\phi}_j' - \bar{A}_{48}^{ij} \bar{\phi}_j = \frac{d\bar{N}_y^{Pi}}{dy} - \bar{N}_{xy2}^{Pi} \\
&\delta \bar{\psi}_i : \bar{A}_{73}^{ij} \bar{u}_j'' + (\bar{A}_{71}^{ij} - \bar{A}_{53}^{ij}) \bar{u}_j' - \bar{A}_{51}^{ij} \bar{u}_j + \bar{A}_{72}^{ij} \bar{v}_j'' + (\bar{A}_{74}^{ij} - \bar{A}_{52}^{ij}) \bar{v}_j' - \bar{A}_{54}^{ij} \bar{v}_j + \bar{A}_{77}^{ij} \bar{\psi}_j'' + (\bar{A}_{75}^{ij} - \bar{A}_{57}^{ij}) \bar{\psi}_j' - (\bar{A}_{55}^{ij} + \bar{B}_{33}^{ij}) \bar{\psi}_j \\
&\qquad + \bar{A}_{76}^{ij} \bar{\phi}_j'' + (\bar{A}_{78}^{ij} - \bar{A}_{56}^{ij}) \bar{\phi}_j' - (\bar{A}_{58}^{ij} + \bar{B}_{31}^{ij}) \bar{\phi}_j - \bar{B}_{32}^{ij} \bar{w}_j' - \bar{B}_{34}^{ij} \bar{w}_j = \frac{d\bar{M}_{xy1}^{Pi}}{dy} - \bar{M}_x^{Pi} - \bar{Q}_{x1}^{Pi} \\
&\delta \bar{\phi}_i : \bar{A}_{63}^{ij} \bar{u}_j'' + (\bar{A}_{61}^{ij} - \bar{A}_{83}^{ij}) \bar{u}_j' - \bar{A}_{81}^{ij} \bar{u}_j + \bar{A}_{62}^{ij} \bar{v}_j'' + (\bar{A}_{64}^{ij} - \bar{A}_{82}^{ij}) \bar{v}_j' - \bar{A}_{84}^{ij} \bar{v}_j + \bar{A}_{67}^{ij} \bar{\psi}_j'' + (\bar{A}_{65}^{ij} - \bar{A}_{87}^{ij}) \bar{\psi}_j' - (\bar{A}_{85}^{ij} + \bar{B}_{13}^{ij}) \bar{\psi}_j \\
&\qquad + \bar{A}_{66}^{ij} \bar{\phi}_j'' + (\bar{A}_{68}^{ij} - \bar{A}_{86}^{ij}) \bar{\phi}_j' - (\bar{A}_{88}^{ij} + \bar{B}_{11}^{ij}) \bar{\phi}_j - \bar{B}_{12}^{ij} \bar{w}_j' - \bar{B}_{14}^{ij} \bar{w}_j = \frac{d\bar{M}_y^{Pi}}{dy} - \bar{M}_{xy2}^{Pi} - \bar{Q}_{y1}^{Pi} \\
&\delta \bar{w}_i : \bar{B}_{23}^{ij} \bar{\psi}_j' - \bar{B}_{43}^{ij} \bar{\psi}_j + \bar{B}_{21}^{ij} \bar{\phi}_j' - \bar{B}_{41}^{ij} \bar{\phi}_j + \bar{B}_{22}^{ij} \bar{w}'' + (\bar{B}_{24}^{ij} - \bar{B}_{42}^{ij}) \bar{w}_j' - \bar{B}_{44}^{ij} \bar{w}_j = \frac{d\bar{Q}_{y2}^{Pi}}{dy} - \bar{Q}_{x2}^{Pi} - \bar{q}_i(y)
\end{aligned} \tag{23}$$

3. THE SOLUTION PROCEDURE

Here, we employ the state-space approach [17] to solve the equilibrium equations obtained in the previous section. The linear system of second-order ordinary differential equations (22) can be expressed in the form of single, first-order, matrix differential equation:

$$\{X'\} = [C]\{X\} + \{F\} \tag{24}$$

where the state vector $\{X\}$ is defined as:

$$\begin{aligned}
&\{X_1\} = \{u_j'\},\ \{X_2\} = \{v_j\},\ \{X_3\} = \{u_j\},\ \{X_4\} = \{v_j'\},\ \{X_5\} = \{\psi_j'\},\ \{X_6\} = \{\phi_j\}, \\
&\{X_7\} = \{\psi_j\},\ \{X_8\} = \{\phi_j'\},\ \{X_9\} = \{w_j'\},\ \{X_{10}\} = \{w_j\}.
\end{aligned} \tag{25}$$

In order to solve equation (24), we assume that $\bar{u}_i(y)$, $\bar{u}_i'(y)$,…, $\bar{w}_i'(y)$ are chosen so that the boundary conditions at $y=0,b$ are identically satisfied. Next, the coefficients A_{mn}^{ij} and B_{mn}^{ij} are found from equations (18). Since these coefficients are constant, the general solution of equation (24) is given by [18]:

$$\{X\} = [U][Q]\{K\} + [U][Q]\int_x [Q]^{-1}[U]^{-1}\{F\}dx \tag{26}$$

where $[U]$ is the matrix of distinct eigenvectors of matrix $[C]$ and $\{K\}$ is a vector of unknown constants to be found by imposing the boundary conditions at edges $x=0,a$. Also, the diagonal matrix $[Q]$ is defined as:

$$[Q] = diag\,(e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_{10n} x}) \tag{27}$$

where λ_k $(k = 1,2,\ldots,10n)$ are the eigenvalues associated with matrix $[C]$.

Next, we can substitute the general solution of $u_i(y)$, $u_i'(y)$,..., $w_i'(y)$ into equations (20) to find $\bar{A}_{mn}^{ij}$ and $\bar{B}_{mn}^{ij}$ which, here, will be constant. The solution procedure for equations (23) is analogous to the one presented for equations (22) and therefore, for the sake of brevity will not be taken up here. Solving the coupled systems of ordinary differential equations will be continued until the solution is converged.

4. RESULTS AND DISCUSSION

In this section, to demonstrate the accuracy and validity of the present method an antisymmetric angle-ply square laminate [p/-45°/30°/45°/-45°/-30°/45°/p] with width-to-thickness ratio b/h=20 will be studied. It is made up of six layers of S-glass/epoxy (passive) and two piezoelectric (active) layers of PZT-4 bonded to the top and bottom surfaces of the passive layers (the material properties of S-glass/epoxy [19] and PZT-4 [20] in the principal material coordinate system are listed in table 1). The passive layers of the hybrid laminate have equal thicknesses while the thicknesses of the active layers are half of each passive layer. A uniform electric potential $\Phi = \Phi_0$ is applied to the upper and lower surfaces of the plate, with the other surfaces of piezoelectric layers grounded.

Property	E_1 (GPa)	E_2	G_{12}	G_{23}	G_{31}	υ_{12}	e_{31} (cm^{-2})	e_{32}	e_{24}
S-glass/epoxy	55	16	7.6	7.6	7.6	0.28	-	-	-
PZT-4	81.3	81.3	30.6	25.6	25.6	0.329	-5.2	-5.2	12.72

Table 1 : The material properties of S-glass/epoxy [19] and PZT-4 [20].

To show the boundary conditions on the four edges of the plate a 4-word notation such as SFSC is employed, in which "S" denotes simply supported, "C" clamped, and "F" free boundary conditions. The 1-4th word indicates the boundary conditions on edges x=0, y=0, x=a, and y=b respectively. It is to be noted that the simply supported boundary conditions at the edges of the laminate are defined as:

$$u_i = w_i = \phi_i = N_{xy}^i = M_x^i = 0 \quad \text{at} \quad x=0,a \tag{28a}$$

$$v_i = w_i = \psi_i = N_{xy}^i = M_y^i = 0 \quad \text{at} \quad y=0,a \tag{28b}$$

In the numerical results the non-dimensionalized variables are deflection $\bar{w} = wE_p/(e_{32}\Phi_0)$, in-plane stresses $(\bar{\sigma}_x, \bar{\sigma}_y, \bar{\sigma}_{xy}) = (\sigma_x, \sigma_y, \sigma_{xy})b/(e_{32}\Phi_0)\times 10^{-2}$, and transverse shear stresses $(\bar{\sigma}_{xz}, \bar{\sigma}_{xz}) = (\sigma_{xz}, \sigma_{xz})b/(e_{32}\Phi_0)$. Here, E_p denotes the Young's modulus of the piezoelectric layer.

The present numerical results will be compared with those obtained from levy-type solutions. As a benchmark, a Levy-type solution based on FSDT is developed for the analysis of piezoelectric laminated plates. It is well known that Levy's solution exists only for cross-ply and antisymmetric angle-ply laminates with two opposite edges simply supported.

The variation of deflection at (x/a,b/2) corresponding to three sets of SCSC, SFSC, and SSSF boundary conditions presented in figure 2, shows an excellent agreement between the present results and those obtained by Levy's solution. In order to demonstrate the capability of the method to analyze conditions for which there exist no Levy-type solutions, deflection of CCCS plate is also depicted in figure 2. As it is expected, the curve corresponding to boundary conditions CCCS is located above the other curves.

Figures 3-5 illustrate, respectively, the through-thickness distributions of normal stress $\bar{\sigma}_x(a/2, b/2, z/h)$ and transverse shear stresses $\bar{\sigma}_{xz}(a/4, b/4, z/h)$ and $\bar{\sigma}_{yz}(a/4, b/4, z/h)$ for different sets of boundary conditions (the numerical values of interlaminar stresses are obtained by integrating the local equilibrium equations of elasticity).

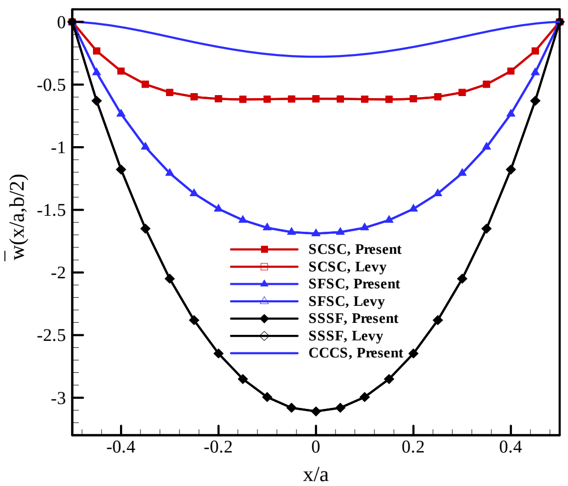

Figure 2 : Variations of deflection versus *x*/*a*.

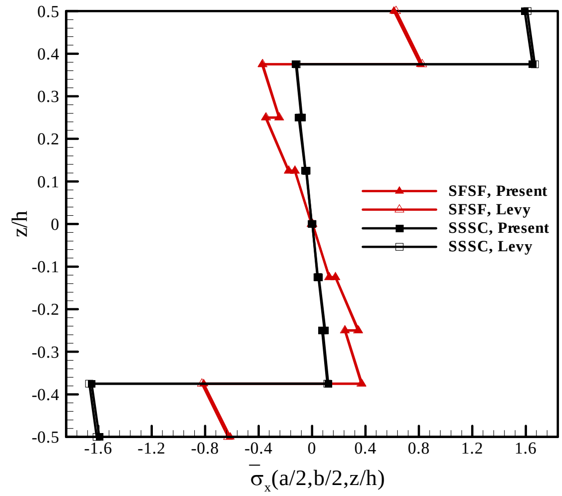

Figure 3 : Through-thickness distributions of normal stress $\bar{\sigma}_x\,(a/2, b/2, z/h)$.

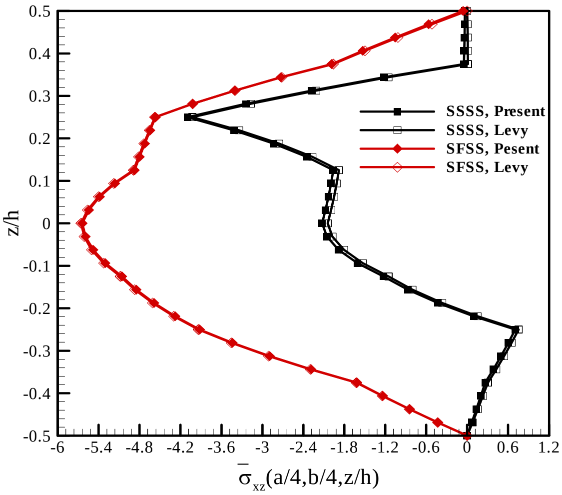

Figure 4 : Through-thickness distributions of transverse shear stress $\bar{\sigma}_{xz}\,(a/4, b/4, z/h)$.

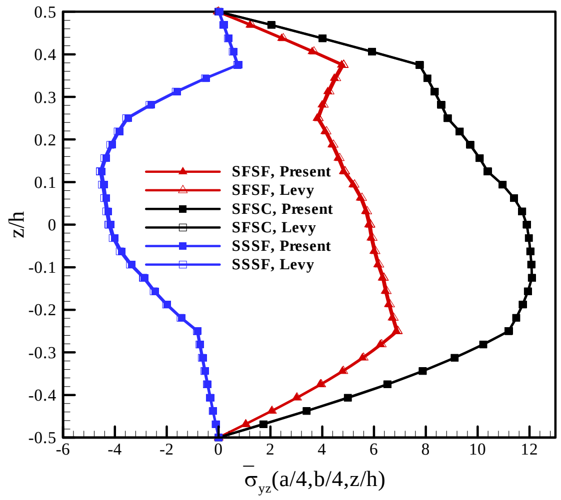

Figure 5 : Through-thickness distributions of transverse shear stress $\bar{\sigma}_{yz}\,(a/4, b/4, z/h)$.

Again, it can be seen that there are close agreements between the results obtained by the two methods. However, it can be said that the magnitude of errors depends on the type of boundary conditions imposed on the edges of the plate.
Table 2 attempts to show the influence of the value of *n* (the total number of summed terms in equation (1)) on the preciseness of the numerical results obtained from the present method. The numerical values of deflection $\bar{w}$ and stress $\bar{\sigma}_{xz}$ listed in table 2 have been obtained for the described laminated plate under SFSF boundary conditions.

n	1	2	3	4	5	Levy's solution
$\bar{w}(a/2, b/2)$	-4.1228	-4.1393	-4.1435	-4.1441	-4.1441	-4.1441
$\bar{\sigma}_{xz}\,(a/4, b/4, 0)$	-6.1662	-9.2579	-8.4799	-8.1129	-8.0304	-8.0826

Table 2 : Non-dimensionalized deflection and transverse shear stress versus *n*.

Table 2 indicates that as the number n is increased, the accuracy of the results is also increased. However, it is observed that the rate of convergence of stress to the result of Levy-type solution is slower compared with that of deflection. It is to be noted that all the numerical results in this study have been achieved with $n = 5$.

5. CONCLUSIONS

In this work, an analytical method based on the FSDT is proposed for bending analysis of piezoelectric laminated plates subjected to arbitrary boundary conditions at the edges. The numerical results are obtained for an antisymmetric angle-ply piezoelectric laminate and compared with those obtained from Levy's solution. It is found that the theory can predict accurately displacements and stresses of the piezoelectric laminated plates. Also a convergence study is performed to determine suitable number of summed terms in the displacement field. Numerical results clearly indicate that by using five terms accurate results are obtained for the deflection and stresses.

REFERENCES

1. **Crawley, E.** and **de Luis, J.,** "Use of piezoceramic actuators as elements of intelligent structures," *AIAA J.*, **25** (1987), 1373–85.
2. **Lee, C.K.** and **Moon F.C.,** "Laminated piezopolymer plates for torsion and bending sensors and actuators," *J. Acoust. Soc. Am.*, **85**/6 (1989), 2432-39.
3. **Lee, C.K.,** "Theory of laminated piezoelectric plates for the design of distributed sensors/ actuators, Part I: Governing equations and reciprocal relationship," *Acoust. Soc. Am.*, **87**/3 (1990), 1144–1158.
4. **Wang B.T.** and **Rogers C.A.**, "Laminate plate theory for spatially distributed induced strain actuators," *J. Compos Mater*, **25**/4 (1991), 433–52.
5. **Jonnalagadda K.D., Blandford G.E.** and **Tauchert T.R.,** "Piezothermoelastic composite plate analysis using first-order shear deformation theory," *Comput. Struct.*, **51** (1994), 79-89.
6. **Mitchell J.A.** and **Reddy J.N.,** "A refined hybrid plate theory for composite laminates with piezoelectric laminates with piezoelectric laminate," *Int. J. Solids Struct.*, **32**/16 (1995), 2345–67.
7. **Huang J.H.** and **Wu T.,** "Analysis of hybrid multilayered piezoelectric plates," *Int. J. Eng. Sci.*, **34** (1996), 171–81.
8. **Kapuria S., Dube G.P., Dumir P.C.** and **Sengupta S.,** "Levy-type piezothermoelastic solution for hybrid plate by using first-order shear deformation theory," *Composites Part* B, **28B** (1997), 535-46.
9. **Zhang, X.D.** and **Sun, C.T.,** "Analysis of a sandwich plate containing a piezoelectric core," *Smart Mater. Struct.*, **8** (1999), 31–40.
10. **S. S. Vel** and **R. C. Batra,** "Cylindrical bending of laminated plates with distributed and segmented piezoelectric actuators/sensors," *AIAA J.*, **38** (2000), 857–67.
11. **S. S. Vel** and **R. C. Batra,** "Generalized plane strain thermoelastic deformation of laminated anisotropic thick plates," *Int. J. Solids Struct.*, **38** (2001), 1395–1414.
12. **S. S. Vel** and **R. C. Batra,** "Generalized plane strain thermopiezoelectric analysis of multilayered plates," *J. Thermal Stresses*, **26** (2003), 353–77.
13. **Zhang C., Cheung Y.K., Di S.** and **Zhang N.,** "The exact solution of coupled thermoelectroelastic behavior of piezoelectric laminates," *Comput. Struct.*, **80** (2002), 1201–12.
14. **Cheng J., Wang B.** and **Du Sh.,** "A theoretical analysis of piezoelectric/composite laminate with larger-amplitude deflection effect, Part I: Fundamental equations," *Int. J. Solids Struct.*, **42** (2005), 6166–6180.
15. **Reddy, J.N.,** "Mechanics of laminated composite plates and shells," 2nd ed., CRC Press (2004).
16. **Fung, Y.C.,** "Foundation of solid mechanics," 1st ed., Englewood Cliffs, New Jersey, Prentice-Hall (1965).
17. **Goldberg, J.L.** and **Schwartz, A.J.,** "Systems of ordinary differential equations," An Introduction. New York, Harper and Row (1972).
18. **Franklin, J.N.,** "Matrix theory," Englewood Cliffs, New Jersey, Prentice-Hall (1968).
19. **Victor M.F.C., Cristovao M.M.S.** and **Carlos A.M.S.,** "Buckling optimization of composite laminated adaptive structures," *Compos. Struct.*, **62** (2003), 315–21.
20. **Fung R.F., Huang J.S.** and **Jan S.C.,** "Dynamic analysis of a piezothermoelastic resonator with various shapes," *ASME J. Vib. Acoust.*, **122** (2000), 244–53.