ANALYSIS OF FUNCTIONALLY GRADED CYLINDRICAL SHELLS SUBJECTED TO MECHANICAL AND THERMAL LOADINGS

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ABSTRACT

Theoretical formulation based on the first-order shear deformation shell theory (FSDST) is presented for the analysis of functionally graded (FG) cylindrical shells subjected to axisymmetric thermomechanical loadings. Also a cylinder made up of a FG layer at the mid-depth and two equal-thickness ceramic and metal layers at the inner and outer surfaces of the cylinder, respectively, is analyzed. The axisymmetric heat transfer equation and the governing equilibrium equations are solved analytically. To check the correctness and accuracy of the present method, the results achieved from this theory are compared with those obtained by utilizing a commercial finite element package. The results show that there are close agreements between the present solutions and those obtained from the finite element method. Finally, the effect of the imposed thermomechanical loading on the response of the FG cylindrical shell is discussed.

1. INTRODUCTION

In conventional laminated composite material, there is a high chance that debonding will occur at some extreme loading conditions. On the other hand, gradually varying the volume fraction of the constituents rather than abruptly changing them over an interface can resolve this problem. Functionally graded materials (FGMs) are composite materials which exhibit a progressive change in composition, structure, and properties as a function of spatial direction within the material. This is achieved by gradually varying the volume fraction of the constituent materials. By spatially varying the microstructure, the material can be tailored for a particular application to yield optimal thermal and mechanical behavior. To this end, in recent years, these types of advanced materials are gradually being used in many engineering applications. Thin-walled members, which are used in reactor vessels, turbines, and other machine parts, are some applications of FGMs in thermomechanical loading conditions.

Many studies for thermoelastic analysis of FGM plates and shells are available in literatures. Reddy and Chin [1] considered thermoelastic analysis, including the coupling effect, for FGM plates and cylinders. Obata and Noda [2] studied the thermal stresses in a FGM hollow sphere and in a hollow circular cylinder. Praveen et al. [3] used the finite element formulation of axisymmetric heat transfer equation to analyze a FG ceramic-metal cylinder. Using the Frobenius series method, Zimmerman and Lutz [4] investigated circular cylinders subjected to a uniform heating. Liew et al. [5] analyzed the thermomechanical behavior of hollow circular cylinders of FGM by using a limiting process that employs the solutions of homogeneous hollow circular cylinders, with no recourse to the basic theory or the equations of non-homogeneous thermoelasticity. Tarn [6], using state space method, analyzed the temperature fields and stress fields of a FGM cylindrical shell with the material constants being a particular power function of the radial variable.

Tutuncu and Ozturk [7] obtained closed-form solutions for stresses and displacement in FG cylindrical and spherical vessels subjected to internal pressure alone using the infinitesimal theory of elasticity.

In this study, based on the first-order shear deformation shell theory, functionally graded cylindrical shells subjected to thermomechanical loadings are analyzed. It is assumed that the loadings are axisymmetric. Various loading and boundary conditions are considered and also FG cylinders are compared with cylindrical shells made up of a FG layer at the mid-depth and two equal-thickness ceramic and metal layers at the inner and outer surfaces of the shells, respectively. To check the correctness and accuracy of the present method, the present results are compared with those obtained by utilizing the finite element method.

2. THEORETICAL FORMULATION

Consider a FG cylindrical shell of length *L*, total thickness *h*, and radius *R*, which is made from a mixture of ceramics and metals (see figure 1). It is assumed that the material is isotropic, and the grading is assumed to be only through the thickness. It is further assumed that the cylindrical shell is subjected to an axisymmetric internal pressure or an axisymmetric thermal loading. The deformations, defined with reference to a coordinate system (*x*,*y*,*z*), taken at the middle surface, are *u*, *v*, and *w* in the *x*, *y* (or θ), and *z* directions, respectively.

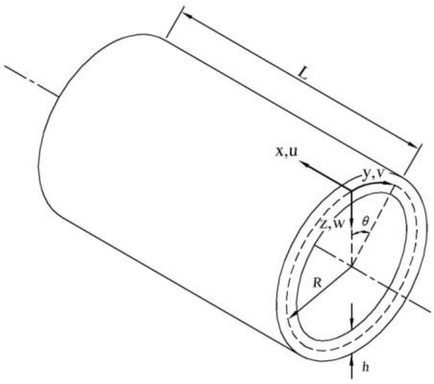


Figure 1 : Geometry and coordinate system of a cylindrical shell.

2.1- Displacement field and strains

Here the cylindrical shell will be studied within the framework of the first-order shear deformation shell theory. The displacement field of the cylinder in axisymmetric loading conditions is independent of coordinate *y* and, therefore, may be represented as:

$$\begin{aligned} &u(x,y,z)=u_0(x)+z\,\psi_x(x), \quad v(x,y,z)=v_0(x)+z\,\psi_y(x) \\ &w(x,y,z)=w(x) \end{aligned} \tag{1}$$

where u_0, v_0, and *w* denote the displacements of a point on the middle surface of the cylinder (*z*=0). Also ψ_x and ψ_y are the rotation functions of the transverse normals on the plane *z*=0. The linear strain-displacement relations of elasticity in cylindrical coordinates are given by [8]:

$$\begin{aligned} &\varepsilon_x=\frac{\partial u}{\partial x}, \quad \varepsilon_y=\frac{1}{1+z/R}\left(\frac{\partial v}{\partial y}+\frac{w}{R}\right), \quad \varepsilon_z=\frac{\partial w}{\partial z}, \quad \gamma_{xy}=\frac{\partial v}{\partial x}+\frac{1}{1+z/R}\frac{\partial u}{\partial y} \\ &\gamma_{yz}=\frac{1}{1+z/R}\left[\frac{\partial w}{\partial y}-\frac{v}{R}+\left(1+z/R\right)\frac{\partial v}{\partial z}\right], \quad \gamma_{xz}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} \end{aligned} \tag{2}$$

Upon substitution of equations (1) into equations (2) and by assuming Love's shell theory [9] (i.e., $1+z/R\cong 1$) the following results will be obtained:

$$\varepsilon_x = \frac{\partial u_0}{\partial x} + z\frac{\partial \psi_x}{\partial x}, \quad \varepsilon_y = \frac{w}{R}, \quad \varepsilon_z = 0, \quad \gamma_{xy} = \frac{\partial v_0}{\partial x} + z\frac{\partial \psi_x}{\partial x}$$
$$\gamma_{yz} = \psi_y - \frac{v_0}{R}, \quad \gamma_{xz} = \psi_x + \frac{\partial w}{\partial x} \tag{3}$$

2.2- Constitutive relations

Here we assume that the material property gradation is through the thickness of the shell. For a shell with a uniform thickness *h* and a reference surface at its middle surface, the volume fraction can be written as the following power-law expression:

$$V_f = \left(\frac{z}{h} + \frac{1}{2}\right)^n \tag{4}$$

where *n* ($0 \le n \le \infty$) is a parameter that dictate the material variation profile through the thickness. For a functionally graded solid with two constituent materials, the variation of material properties can be expressed as:

$$p(z) = (p_i - p_o)V_f + p_o \tag{5}$$

where *p* denotes a generic material property like modulus, p_o and p_i denote the property of the outer and inner surfaces of the cylinder, respectively. Here we assume that moduli *E* and *G*, coefficient of thermal expansion α, and thermal conductivity *k* vary according to equation (5) and the Poisson's ratio ν is assumed to be a constant.
The linear constitutive relations are:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \alpha \Delta T \right), \quad \begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{6}$$

where

$$Q_{11} = Q_{22} = \frac{E(z)}{1-\nu^2}, \quad Q_{12} = \frac{\nu E(z)}{1-\nu^2}, \quad Q_{44} = Q_{55} = Q_{66} = \frac{E(z)}{2(1+\nu)} = G(z) \tag{7}$$

and ΔT is the temperature change from a stress-free state that will be obtained by solving the one-dimensional heat transfer equation.

2.3- Equations of equilibrium

Using the principle of minimum total potential energy [8], the equilibrium equations can be shown to be:

$$\delta u_0 : \frac{dN_x}{dx} = 0, \qquad \delta v_0 : \frac{dN_{xy}}{dx} + \frac{Q_y}{R} = 0, \qquad \delta w : \frac{dQ_x}{dx} - \frac{N_y}{R} + P_z(x) = 0$$
$$\delta \psi_x : \frac{dM_x}{dx} - Q_x = 0, \quad \delta \psi_y : \frac{dM_{xy}}{dx} - Q_y = 0 \tag{8}$$

where P_z is the internal pressure applied to the cylinder. In equations (8) the force and moment resultants are defined as:

$$\left(N_x, N_y, N_{xy}\right) = \int_{-h/2}^{h/2} \left(\sigma_x, \sigma_y, \sigma_{xy}\right) dz$$
$$\left(M_x, M_y, M_{xy}\right) = \int_{-h/2}^{h/2} \left(\sigma_x, \sigma_y, \sigma_{xy}\right) z\,dz, \quad \left(Q_x, Q_y\right) = \int_{-h/2}^{h/2} \left(\sigma_{xz}, \sigma_{yz}\right) dz \tag{9}$$

The boundary conditions consist of specifying u_0 or N_x, v_0 or N_{xy}, w or Q_x, ψ_x or M_x, and ψ_y or M_{xy} at x=0 and x=L. Upon substitution of equations (6) into equations (9), the force and moment resultants in terms of displacement components will be obtained which can be presented as follows:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & 0 \\ 0 & 0 & A_{66} & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & 0 \\ 0 & 0 & B_{66} & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} u_0' \\ \dfrac{w}{R} \\ v_0' \\ \psi_x' \\ \psi_y' \end{Bmatrix} - \begin{Bmatrix} N_x^T \\ N_y^T \\ 0 \\ M_x^T \\ M_y^T \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = k^2 \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \psi_y - \dfrac{v_0}{R} \\ w' + \psi_x \end{Bmatrix} \tag{10}$$

where

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \int_{-h/2}^{h/2} Q_{ij}\,(1, z, z^2)\,dz \tag{11}$$

and $k^2\left(=5/6\right)$ is the shear correction factor. The thermal resultants in equations (10) are defined as:

$$\left(N_x^T, M_x^T\right) = \int_{-h/2}^{h/2} (Q_{11}+Q_{12})\alpha\Delta T\,(1, z)\,dz \ , \quad \left(N_y^T, M_y^T\right) = \int_{-h/2}^{h/2} (Q_{12}+Q_{22})\alpha\Delta T\,(1, z)\,dz \tag{12}$$

Lastly, the governing equations of equilibrium are obtained by substituting equations (10) into equations (8):

$$\begin{aligned} & A_{11}u_0'' + A_{12}\, w'/R + B_{11}\psi_x'' = dN_x^T/dx \\ & A_{66}v_0'' - k^2A_{44}v_0/R^2 + B_{66}\psi_y'' + k^2A_{44}\psi_y/R = 0 \\ & -A_{12}u_0'/R + k^2A_{55}w'' - A_{22}w/R^2 + (k^2A_{55} - B_{12}/R)\psi_x' = -N_y^T/R - P_z \\ & B_{11}u_0'' + (B_{12}/R - k^2A_{55})w' + D_{11}\psi_x'' - k^2A_{55}\psi_x = dM_x^T/dx \\ & B_{66}v_0'' + k^2A_{44}v_0/R + D_{66}\psi_y'' - k^2A_{44}\psi_y = 0 \end{aligned} \tag{13}$$

The above equations are five coupled second-order ordinary differential equations with constant coefficients that can be solved for any arbitrary boundary conditions. For the sake of brevity, the solution procedure of these equations will not be taken up here.

In order to solve equations (13) in thermal loadings the temperature field should be known. It is assumed that one value of temperature is imposed on the inner surface and the other value on the outer surface of the cylinder. In this case, the temperature distribution through the thickness can be obtained by solving a simple steady state heat transfer equation through the thickness of the cylinder. This equation is given by:

$$-\frac{d}{dz}\left(k(z)\frac{dT}{dz}\right) = 0 \tag{14}$$

and the boundary conditions are T=T_o at z=-h/2 and T=T_i at z=h/2. It is readily seen that the solution to equation (14) is:

$$T\left(\xi\right) = c_{1n}hA_n\left(\xi\right)/\left(k_i - k_o\right) + c_{2n} \tag{15}$$

where

$$\xi = \frac{z}{h} + \frac{1}{2}, \quad A_n(\xi) = \int \frac{d\xi}{\xi^n - \mu^n}$$

$$c_{1n} = \frac{(T_i - T_o)(k_i - k_o)}{h[A_n(1) - A_n(0)]}, \quad c_{2n} = \frac{T_o A_n(1) - T_i A_n(0)}{A_n(1) - A_n(0)} \tag{16}$$

with $\mu^n = k_o/(k_o - k_i)$. It is to be noted that the integral of $A_n(\xi)$ (see equations (16)) has analytical solution for n=0.2, n=0.5, and all integer values. For other values of n, this integral must be solved numerically.

3. NUMERICAL RESULTS

Here we present some representative results for a FG cylindrical shell of thickness h=1cm, radius R=50h, and length L=2R. The cylinder is assumed to be, for example, simply supported and subjected to thermomechanical loadings. The boundary conditions at *x=0, L* of the cylinder are $N_x = PR/2$ and $N_{xy} = M_x = M_{xy} = w = 0$. It is also assumed that the inner surface of the cylinder is rich of ceramic (Zirconia) and the outer surface is rich of metal (Aluminum). The thermomechanical properties of Zirconia and Aluminum are as follows:

$$\begin{aligned} &E_c = 151\,\text{GPa}, \quad \nu_c = 0.3, \quad \alpha_c = 10\times10^{-6}\,/^\circ\text{C}, \quad k_c = 2.09\,\text{W/mK} \\ &E_m = 70\,\text{GPa}, \quad \nu_m = 0.3, \quad \alpha_m = 23\times10^{-6}\,/^\circ\text{C}, \quad k_m = 204\,\text{W/mK} \end{aligned} \tag{17}$$

Figure 2 shows the distribution of the volume fraction V_f of the ceramic phase through the cylinder thickness for various values of the power-law index n.

In what follows, several numerical examples are presented for a cylinder subjected to a uniform internal pressure or a steady state temperature field.

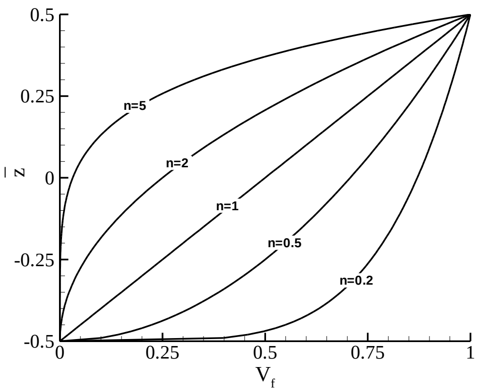


Figure 2 : Variation of the volume fraction V_f of the ceramic phase through the thickness of the FG cylindrical shell.

3.1- Mechanical loading

Consider a cylindrical shell subjected to a uniform internal pressure. In this case, results are presented in terms of non-dimensional variables length $\bar{x} = x/L$, thickness $\bar{z} = z/h$, deflection $\bar{w} = w/R$, and in-plane stresses $(\bar{\sigma}_x, \bar{\sigma}_y) = (\sigma_x, \sigma_y)/p$. The particular problem, which shows the validity and accuracy of the present results, is a FG cylindrical shell with n=1 subjected to a uniform internal pressure of P_z=1MPa. Here the present results are compared with those obtained by utilizing the finite element package of ANSYS.

Figures 3a and 3b show the distributions of non-dimensionalized radial displacement $\bar{w}$, longitudinal stress $\bar{\sigma}_x$, and circumferential stress $\bar{\sigma}_y$ along the length of the cylinder. It is seen that there is a good agreement between the present results and those obtained from finite element method.

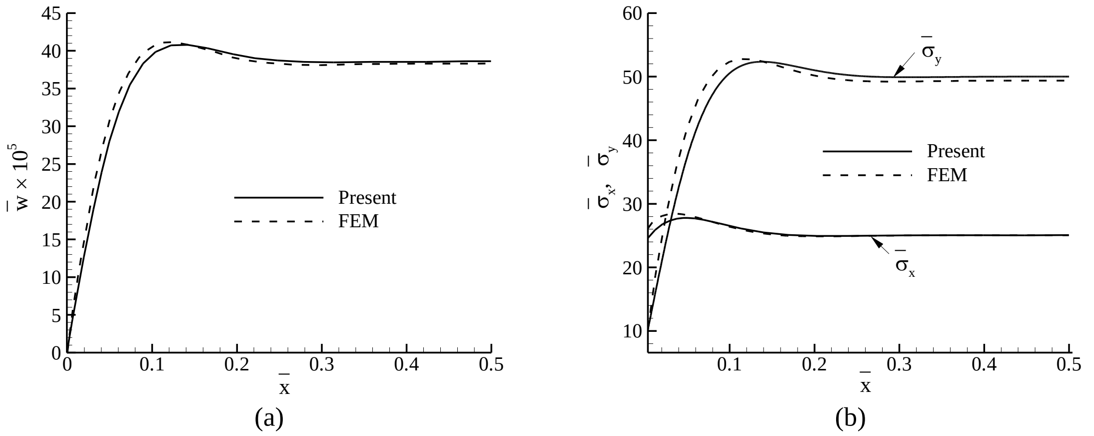


Figure 3 : Distributions of (a) deflection $\bar{w}$ and (b) stresses $\bar{\sigma}_x$ and $\bar{\sigma}_y$ versus $\bar{x}$ of the FG cylindrical shell (n=1) due to internal pressure of P_z=1 MPa.

The variations of $\bar{\sigma}_x$ and $\bar{\sigma}_y$ along the length of the FG cylindrical shell for various values of the power-law index n are shown in figures 4a and 4b, respectively. It is found that these stresses are minimum for n=0.2 and maximum for the cylinders made up of Zirconia or Aluminum.

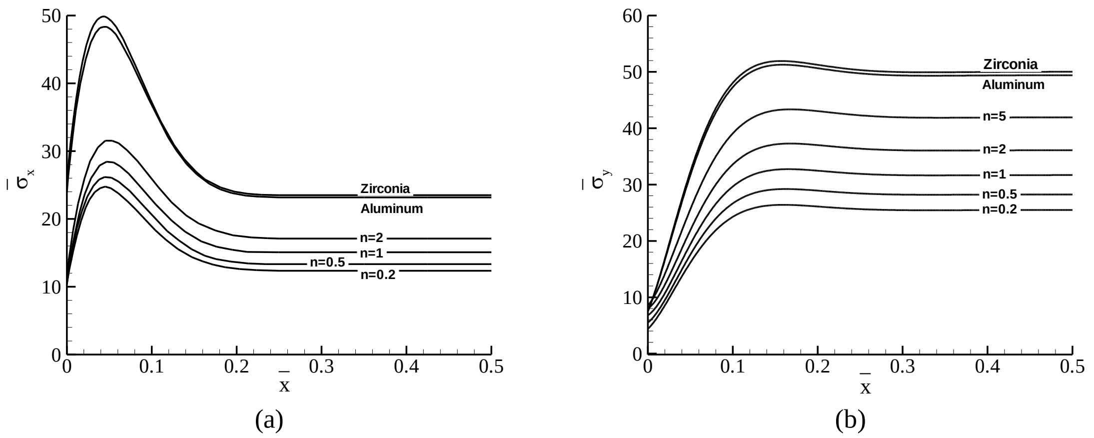


Figure 4 : Distributions of (a) longitudinal stress $\bar{\sigma}_x$ and (b) circumferential stress $\bar{\sigma}_y$ versus $\bar{x}$ at the outer surface of the FG cylindrical shell due to the internal pressure P_z.

Also we consider a shell made up of a FG layer at the mid-depth and two equal-thickness ceramic (Zirconia) and metal (Aluminum) layers at the inner and outer surfaces of the shell, respectively (see figure 5).

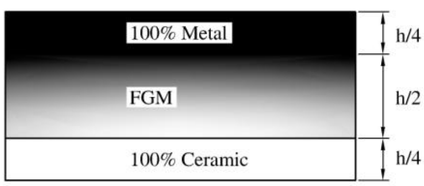
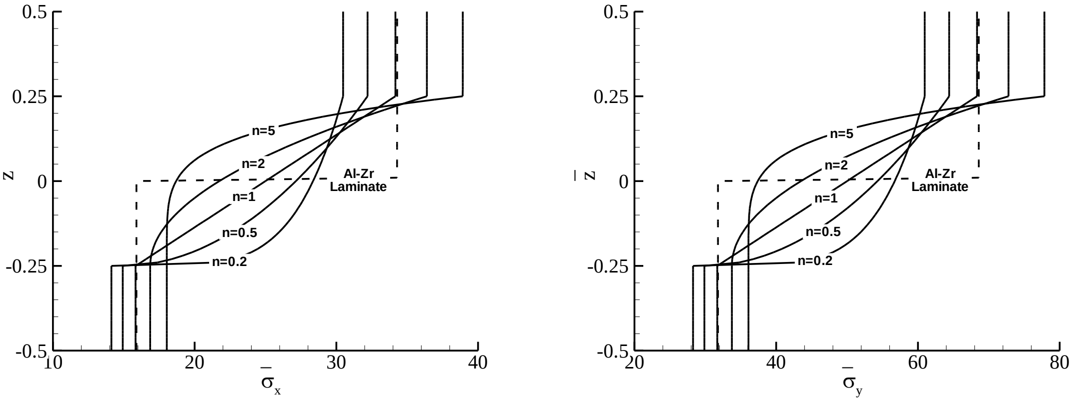


Figure 5 : Thickness of a cylinder made up of a FG layer at the mid-depth and two equal-thickness ceramic and metal layers at the inner and outer surfaces, respectively.

Figures 6a and 6b show distributions of non-dimensionalized stresses $\bar{\sigma}_x$ and $\bar{\sigma}_y$

through the thickness of the FG cylindrical shell and also through the thickness of a conventional laminated Aluminum-Zirconia cylindrical shell. It is seen that in the absence of FGM layer, the stress distributions are discontinuous. It is also observed that by using a FGM layer with $n<1$, we can reduce magnitude of the axial and circumferential stresses.



(a) (b)

Figure 6 : Distributions of (a) longitudinal stress $\bar{\sigma}_x$ and (b) circumferential stress $\bar{\sigma}_y$ through the thickness of the FG cylindrical shell and the two-layer laminated cylinder at $x=L/2$ due to the internal pressure P_z .

3.2- Thermal loading

The FG cylindrical shell is studied under a thermal gradient through its thickness direction. The temperature of the inner ceramic-rich surface is fixed at $T_i=300^{o}$C and that of the outer metal-rich surface is kept constant at $T_o=20^{o}$C. It is assumed that the reference stress-free temperature 0^{o}C. The temperature field through the thickness of the cylinder can be easily obtained from equation (15). For the thermal loading, the in-plane stresses are non-dimensiuonalized as $\left(\bar{\sigma}_x,\bar{\sigma}_y\right)=10\left(\sigma_x,\sigma_y\right)/\left(E_m\alpha_m T_c\right)$.

Figure 7 shows the variation of the temperature through the thickness of the cylindrical shell for various values of the power-law index *n*.

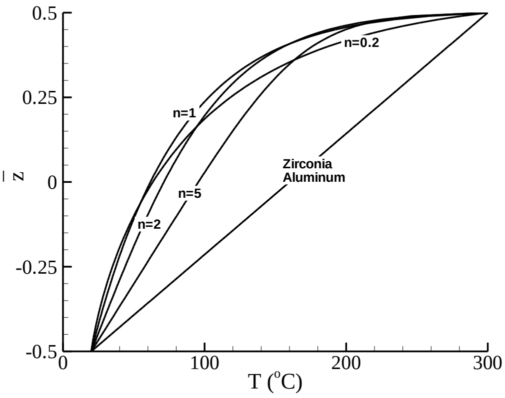
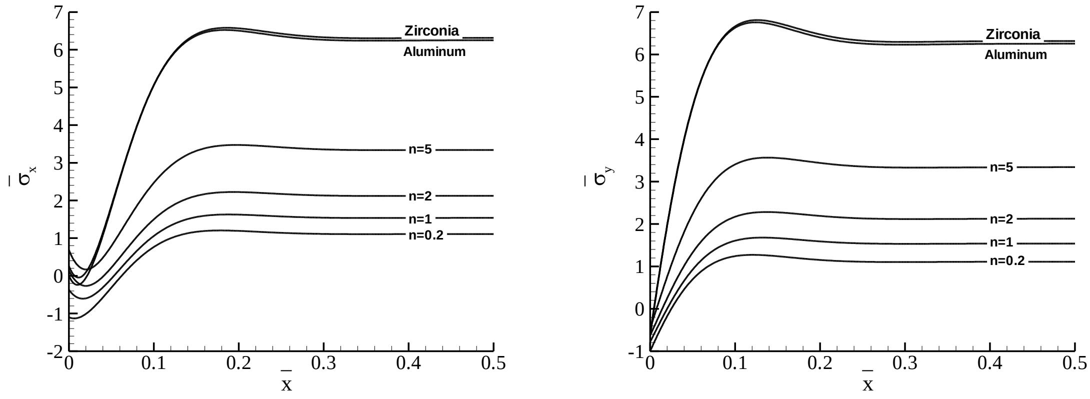


Figure 7 : Temperature profile through the thickness of the FG cylindrical shell.

The distributions of the thermal stresses $\bar{\sigma}_x$ and $\bar{\sigma}_y$ along the length at the outer surface of the FG cylindrical shell for various values of the power-law index *n* are shown in figures 8a and 8b, respectively. It is observed that these stresses are minimum for $n=0.2$ and maximum for cylinders made of Zirconia or Aluminum.



(a) (b)

Figure 8 : Distributions of (a) longitudinal stress $\bar{\sigma}_x$ and (b) circumferential stress $\bar{\sigma}_y$ versus $\bar{x}$ at the outer surface of the FG cylindrical shell due to the thermal loading.

4. CONCLUSIONS

Within the displacement field of a first-order shear deformation shell theory, functionally graded cylindrical shells subjected to axisymmetric thermomechanical loadings are analyzed. Also a bi-material shell made up of a FG layer at the mid-depth and two equal-thickness ceramic and metal layers is analyzed. The effective properties at a point in the shell are assumed to vary according to a power-law distribution in terms of the volume fractions of the constituents. The axisymmetric heat transfer equation and the governing equilibrium equations are solved analytically. The results achieved from this theory are compared with those obtained by utilizing a finite element package. It is found that there are good agreements between the present solutions and those obtained from the finite element method. It is observed that compared to the classical layered composites, FGMs exhibit considerable improvement in the temperature and thermal-stress distributions. This is due primarily to the enforcement of continuity in material properties through the thickness. The present results give us an idea about the type of distribution one should choose in order to control the magnitude of the stresses.

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