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FREQUENCY AND TIME RESPONSE OF ROTORS LONGITUDINAL VIBRATION USING HYBRID MODELING

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Abstract
In this paper, the application of distributed-lumped (hybrid) modeling technique (DLMT), as introduced by Whalley [1], is considered in modeling the forced longitudinal vibration of systems. To illustrate the simplicity and efficiency of the method, an industrial example of rotating shaft with a lumped element subjected to various longitudinal forces is analyzed. Natural frequencies obtained from this method are compared with those obtained by using finite element method. Also time responses of the system subjected to three different types of longitudinal forces are computed by this method.

Keywords: Distributed-Lumped-Modeling-Frequency response-Time response.

1. Introduction
The question of vibration model of industrial systems, especially rotating shafts, is the basic consideration in engineering design of dynamic systems. Not only the avoidance from natural frequencies of such systems have been observed since long before, but the condition monitoring (CM) of highly sensitive and precious plants such as turbines are applied widely using accurate vibration model of systems. Condition Monitoring for rotating machinery incorporates a wide range of techniques, such as oil analysis, wear-debris analysis, ultrasonic, corrosion, and vibration analysis. Vibration condition monitoring is, arguably, the oldest type of machinery condition monitoring. Measured vibration signals can reveal important and detailed information about possible fault which may exist in a machine [2]. Fault identification in rotating machinery using vibration analysis is a constantly expanding field. Developments are continually made with the use of new analysis methods, increased computing power, measurement techniques, and so on.
Among different methods of modeling systems such as lumped-lumped modeling technique (LLMT) and distributed-lumped modeling technique (DLMT), or numerical and approximate methods such as transfer matrix method (TMM) and finite element method (FEM), it is clear that the model combined with both the distributed and lumped elements is the best representative of complex and accurate systems.
Many industrial systems can be modeled as a rotating shaft with disks on it, such as gear systems, propellers, pumps, turbines, mills, etc. In such systems, the disks, which is the representative of blades, gears, etc., is impressed by different loads, which affect the vibration of system and lead to different frequency and time responses. Comparison between safe and defected system responses brings us an effective and advantageous method to the condition monitoring of expensive and important systems such as turbines.
In this study, for a simple example, the longitudinal vibration of a general two–stage distributed-lumped-distributed system is considered. The system is modeled by distributed-lumped technique, and the natural frequencies are investigated for two sets of boundary conditions (B.C.'s): clamped-free and clamped-clamped shaft, which are more common in real systems. To check the correctness and accuracy of the present method, the natural frequencies and mode shapes of an industrial example of the system

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achieved from this method are compared with those obtained by utilizing the commercial finite element package of ANSYS, revision 7. The frequency responses are computed in response to the limited step, limited ramp, and delta force functions, using hybrid model of the system for clamped-free shaft. Time responses are also calculated employing the inverse Fourier transform (IFT) and convolution integral together.

2. The General Distributed-Lumped Model

Generally speaking, hybrid modeling technique deals with systems by dividing them into two element types.

1) The distributed element, which is the main part of shafts, rotors or any other continuous part of the systems with distributed mass or inertia.
2) The lumped element which is the supplementary part of shafts, rotors, etc. with concentrated mass or inertia such as disks, gears, propellers, pulleys, and so on.

In this way, a system is considered as a combined set of distributed and lumped elements, in which the vibration of final model is obtained by setting the distributed and lumped matrices of different parts and combining them together (see Fig. 1). Distributed and lumped matrices are formed according to the analytical equations of motion, so this is the highly accurate method in contrast with other approximate methods such as transfer matrix method, finite element method, and so on. Another advantage of this method compare with analytical method is that the continuity conditions between distributed and lumped elements are identically satisfied and it remains only to satisfy the boundary conditions of the system. To this end, there is no difficulty in using this approach for analyzing systems with mixed series of distributed and lumped elements as shown by Barlett et al. [3].

2.1. Deriving Transfer matrix for Distributed Element

The equations of motion for longitudinal vibration of a thin rod with the density ρ and the modulus of elasticity E can be expressed by the following equations (e.g., see [4,5]):

$$\frac{\partial p(x,t)}{\partial x} = \rho A \frac{\partial^2 u(x,t)}{\partial t^2} \tag{1}$$

$$\frac{\partial u(x,t)}{\partial x} = \frac{p(x,t)}{AE} \tag{2}$$

where $u(x,t)$ and $p(x,t)$ are the displacement and internal force functions, x is the distance along a section and t is time.

Differentiating equation (2) with respect to x and substituting for $\partial p/\partial x$ in equation (1) yields:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 u(x,t)}{\partial t^2} \tag{3}$$

Also differentiating equation (2) twice with respect to t results in:

$$\frac{\partial^3 u(x,t)}{\partial x \partial t^2} = \frac{1}{AE}\frac{\partial^2 p(x,t)}{\partial t^2} \tag{4}$$

Differentiating equation (1) with respect to x gives:

$$\frac{\partial^2 p(x,t)}{\partial x^2} = \rho A \frac{\partial^3 u(x,t)}{\partial x \partial t^2} \tag{5}$$

Substituting equation (4) into (5) results in:

$$\frac{\partial^2 p(x,t)}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 p(x,t)}{\partial t^2} \tag{6}$$

Equations (3) and (6) are the main equations of longitudinal vibration. Next, assuming zero initial conditions, Laplace transformation of equations (6) and (3) gives:

$$\begin{aligned} &\frac{\partial^2 p(s,t)}{\partial x^2} - \frac{\rho}{E}s^2 p(s,t) = 0 \\ &\frac{\partial^2 u(s,t)}{\partial x^2} - \frac{\rho}{E}s^2 u(s,t) = 0 \end{aligned} \tag{7}$$

where s is the Laplace transform variable. Equations (7) can be written in compact form as:

$$\frac{\partial^2 k}{\partial x^2} - \Gamma^2 k = 0 \tag{8}$$

where

$$k = u(x,s) \text{ or } p(x,s) \tag{9}$$

and

$$\Gamma = s\sqrt{\frac{\rho}{E}} \tag{10}$$

The general solution of equation (8) is given by:

$$k = r_1 e^{\Gamma x} + r_2 e^{-\Gamma x}$$

where

$$e^{\Gamma x} = \cosh \Gamma x + \sinh \Gamma x$$

$$e^{-\Gamma x} = \cosh \Gamma x - \sinh \Gamma x$$

Therefore,

$$k = (r_1 + r_2)\cosh \Gamma x + (r_1 - r_2)\sinh \Gamma x \tag{11}$$

Hence, the solution of equations (7) will be:

$$\begin{aligned} p(x,s) &= A\cosh \Gamma x + B\sinh \Gamma x \\ u(x,s) &= C\sinh \Gamma x + D\cosh \Gamma x \end{aligned} \tag{12}$$

The unknown constants of integration A and D are obtained by imposing the boundary conditions at x=0. That is,

$$\begin{aligned} A &= p(0,s) \\ D &= u(0,s) \end{aligned} \tag{13}$$

Next, it remains to find B and C in equations (12). Differentiating equation (12) with respect to x and substituting for $\partial p/\partial x$ and $\partial u/\partial x$ from Laplace transformation of equations (1) and (2), respectively, gives:

$$\begin{aligned} \rho A s^2 u(x,s) &= A\Gamma \sinh \Gamma x + B\Gamma \cosh \Gamma x \\ \frac{1}{AE}\, p(x,s) &= C\Gamma \cosh \Gamma x + D\Gamma \sinh \Gamma x \end{aligned} \tag{14}$$

Now putting $x=0$ in equations (14) results in:

$$\begin{aligned} B &= sA\sqrt{\rho E}\; u(0,s) = \xi u(0,s) \\ C &= \frac{1}{sA\sqrt{\rho E}}\, p(0,s) = \frac{1}{\xi}\, p(0,s) \end{aligned} \tag{15}$$

Hence, the solution of equations (7) for the j[th] element can be expressed in matrix form as:

$$\begin{Bmatrix} p_j(x,s) \\ u_j(x,s) \end{Bmatrix} = \begin{bmatrix} \cosh \Gamma_j x & \xi_j \sinh \Gamma_j x \\ \frac{1}{\xi_j} \sinh \Gamma_j x & \cosh \Gamma_j x \end{bmatrix} \begin{Bmatrix} p_j(0,s) \\ u_j(0,s) \end{Bmatrix}$$

where

$$\xi_j = sA_j\sqrt{\rho_j E_j}$$

According to Fig. 1, for the j[th] element at $x=0$

$$p_j(0,s) = p_{j-1}(s)$$

$$u_j(0,s) = u_{j-1}(s)$$

Therefore,

$$\begin{Bmatrix} p_j(s) \\ u_j(s) \end{Bmatrix} = \begin{bmatrix} \cosh \Gamma_j l_j & \xi_j \sinh \Gamma_j l_j \\ \frac{1}{\xi_j} \sinh \Gamma_j l_j & \cosh \Gamma_j l_j \end{bmatrix} \begin{Bmatrix} p_{j-1}(s) \\ u_{j-1}(s) \end{Bmatrix} \tag{16}$$

2.2. Deriving Transfer Matrix for Lumped Element

The equation of motion and continuity conditions in Laplace domain of the j[th] lumped element, which is exposed to the applied force f in the x-direction are written as:

$$\begin{aligned} p_j(s) - p_{j-1}(s) + f_j(s) &= m_j s^2 u_j(s) \\ u_j(s) &= u_{j-1}(s) \end{aligned} \tag{17}$$

Therefore, equation (17) can be expressed in the matrix form as:

$$\begin{Bmatrix} p_j(s) \\ u_j(s) \end{Bmatrix} = \begin{bmatrix} 1 & m_j s^2 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} p_{j-1}(s) \\ u_{j-1}(s) \end{Bmatrix} + \begin{Bmatrix} -f_j \\ 0 \end{Bmatrix} \tag{18}$$

3. Illustrative Example

In this section, the methodology outlined previously is applied to a shaft with a disk on its middle (see Fig. 2), which is a simplified model for very useful and common industrial systems. The specifications of the system considered here are shown in Table 1. As mentioned already, the present method can be used for analyzing systems with any number of distributed and lumped elements without any increasing in difficulty.

3.1. DLMT Solution

To represent the main hybrid model of the system, it should be noticed that the system is combined of two distributed and one lumped elements (Fig. 3). For the distributed elements 1 and 3 the transfer matrices can be written according to equation (16) as:

$$\begin{Bmatrix} p_1 \\ u_1 \end{Bmatrix} = [T_D]_1 \begin{Bmatrix} p_0 \\ u_0 \end{Bmatrix}, \quad \begin{Bmatrix} p_3 \\ u_3 \end{Bmatrix} = [T_D]_3 \begin{Bmatrix} p_2 \\ u_2 \end{Bmatrix} \tag{19}$$

where

$$[T_D] = \begin{bmatrix} \cosh \Gamma l & \xi \sinh \Gamma l \\ \frac{1}{\xi} \sinh \Gamma l & \cosh \Gamma l \end{bmatrix} \tag{20}$$

Also the transfer matrix for the lumped element is:

$$\begin{Bmatrix} p_2 \\ u_2 \end{Bmatrix} = [T_L]_2 \begin{Bmatrix} p_1 \\ u_1 \end{Bmatrix} + \begin{Bmatrix} -f \\ 0 \end{Bmatrix} \tag{21}$$

where

$$[T_L]_2 = \begin{bmatrix} 1 & m_2 s^2 \\ 0 & 1 \end{bmatrix} \tag{22}$$

Substituting equations (19) into (21) yields [6,7]:

$$\begin{Bmatrix} p_3 \\ u_3 \end{Bmatrix} = \begin{bmatrix} \cosh 2\Gamma l + \frac{1}{2} m_2 \xi^{-1} s^2 \sinh 2\Gamma l & \xi \sinh 2\Gamma l + m_2 s^2 \cosh^2 \Gamma l \\ \xi^{-1} \sinh 2\Gamma l + m_2 \xi^{-2} s^2 \sinh^2 \Gamma l & \cosh 2\Gamma l + \frac{1}{2} m_2 \xi^{-1} s^2 \sinh 2\Gamma l \end{bmatrix} \begin{Bmatrix} p_0 \\ u_0 \end{Bmatrix} + \begin{bmatrix} \cosh \Gamma l & \xi \sinh \Gamma l \\ \xi^{-1} \sinh \Gamma l & \cosh \Gamma l \end{bmatrix} \begin{Bmatrix} -f \\ 0 \end{Bmatrix} \tag{23}$$

Equation (23) may be shown in the simple form as:

$$\begin{Bmatrix} p_3 \\ u_3 \end{Bmatrix} = [C] \begin{Bmatrix} p_0 \\ u_0 \end{Bmatrix} + [D] \begin{Bmatrix} -f \\ 0 \end{Bmatrix} \tag{24}$$

where

$$\begin{aligned} [C] &= \begin{bmatrix} \cosh 2\Gamma l + \frac{1}{2} m_2 \xi^{-1} s^2 \sinh 2\Gamma l & \xi \sinh 2\Gamma l + m_2 s^2 \cosh^2 \Gamma l \\ \xi^{-1} \sinh 2\Gamma l + m_2 \xi^{-2} s^2 \sinh^2 \Gamma l & \cosh 2\Gamma l + \frac{1}{2} m_2 \xi^{-1} s^2 \sinh 2\Gamma l \end{bmatrix} \\ [D] &= \begin{bmatrix} \cosh \Gamma l & \xi \sinh \Gamma l \\ \xi^{-1} \sinh \Gamma l & \cosh \Gamma l \end{bmatrix} \end{aligned} \tag{25}$$

Equation (24) is the transfer matrix of the overall system relating axial forces and axial displacements of the left and right end of the system.

The Laplace transform variable ‘s’, in general, is the representative of equation $s = \sigma + i\omega$; in which the real part (σ) shows damping, and the imaginary part ($i\omega$) shows vibrating frequency. It is assumed in the present example that $\sigma = 0$ and, therefore, equation (24) will be altered from Laplace domain into frequency domain by putting $s = i\omega$ [6].

For each sets of boundary conditions one characteristic equation can be obtained that its solutions will give the natural frequencies of the system. In what follows, a rotating shaft with a lumped mass on it with two different boundary conditions will be considered.

3.1.1. Clamped-Free System

Assuming that the shaft is clamped at the position zero, and free at the position 3, the boundary conditions will be (see Fig. 2):

$$\begin{aligned} u_0 &= 0 \quad (at\ clamped\ end) \\ p_3 &= 0 \quad (at\ \ free\ end) \end{aligned} \tag{26}$$

According to the above relations, equation (24) can be arranged as:

$$\begin{Bmatrix} p_0 \\ u_3 \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{C_{11}} & -\dfrac{C_{12}}{C_{11}} \\ \dfrac{C_{21}}{C_{11}} & -\dfrac{C_{21}C_{12}}{C_{11}} + C_{22} \end{bmatrix} \begin{Bmatrix} p_3 \\ u_0 \end{Bmatrix} + \begin{bmatrix} -\dfrac{D_{11}}{D_{11}} & 0 \\ -\dfrac{C_{21}}{C_{11}} D_{11} + D_{21} & 0 \end{bmatrix} \begin{Bmatrix} -f \\ 0 \end{Bmatrix} \tag{27}$$

where u_0, p_3, and f are the inputs and u_3 and p_0 are the outputs of the system.
In this case, the natural frequencies are obtained by plotting u_3/f, for instance, as shown in Fig. 4. In this view, the natural frequencies occur at the peaks of the spectrum. From equation (27), it is clear that the peaks are the result of denominator approaching zero. Since in all relations C_{11} is the denominator, so the natural frequencies are the roots of C_{11}.
Other than that, putting the relations (26) in equation (24), and neglecting the term coincides with f (because the natural frequencies are independent of applied force) gives:

$$\begin{aligned} C_{11} p_0 &= 0 \\ C_{21} p_0 &= u_3 \end{aligned} \tag{28}$$

The first equation satisfies when $C_{11} = 0$, which is another reason for computing the roots of C_{11} to find the natural frequencies as well. The results for this case are listed in Table 2.

3.1.2. Clamped-Clamped System

In this case, the boundary conditions are expressed as:

$$\begin{aligned} u_0 &= 0 \\ u_3 &= 0 \end{aligned} \tag{29}$$

Hence, equation (24) can be arranged as:

$$\begin{Bmatrix} p_3 \\ p_0 \end{Bmatrix} = \begin{bmatrix} \dfrac{C_{11}}{C_{21}} & -\dfrac{C_{11}C_{22}}{C_{21}} + C_{12} \\ \dfrac{1}{C_{21}} & -\dfrac{C_{22}}{C_{21}} \end{bmatrix} \begin{Bmatrix} u_3 \\ u_0 \end{Bmatrix} + \begin{bmatrix} D_{11} - D_{21}\dfrac{C_{11}}{C_{21}} & 0 \\ -\dfrac{D_{21}}{C_{21}} & 0 \end{bmatrix} \begin{Bmatrix} -f \\ 0 \end{Bmatrix} \tag{30}$$

where u_0, u_3, and f are the inputs and p_3 and p_0 are the outputs of system. The natural frequencies are obtained by plotting p_3/f, for instance, as shown in Fig. 5. Similar to the discussion presented in the previous part, the roots of C_{21} should be computed in this case. The results for this case are listed in Table 3.

3.2. FEM Solution

To contrast and confirm the results with another method, the finite element method is used to investigate the natural frequencies. The system is modeled by ANSYS (7) software, and meshed using brick 45 (8 nodes 3D) elements. Block Lanczos solver of ANSYS is used in the analysis. The natural frequencies are listed in Tables 2 and 3. Also the first two mode shapes for clamped-free and clamped-clamped boundary conditions are shown in Figs. 6-9.

3.3. Frequency and Time Responses

Since the rotor systems are usually subjected to different external forces, the effect of three important types of forces on a clamped-free shaft is investigated.
To acquire the frequency response, three types of applied forces (limited step, limited ramp and impulse functions) in Laplace form are substituted into equation (24). These three types of forces are shown in Figs. 10-12. The frequency responses for u_3 and p_0 showed that these parameters tend to infinity at the natural frequencies. Also the disk displacement can be obtained through equation (21). For the sake of brevity, however, theses parameters are not presented here.
To compute the time response, both the inverse Fourier transform and convolution integral are used together. Firstly, the inverse Fourier transform which is expressed as [8]:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} x(\omega) e^{i\omega t}\, d\omega \tag{31}$$

is used to find the time response of system to the unit impulse (without delay). In equation (31), $x(\omega)$ is p_0/f, u_3/f or u_1/f which are obtained from equations (27) or (21). Since the referred functions are complex, the integral relation can be expressed numerically as:

$$x(t) = \frac{1}{\pi} \sum_{r=1}^{k} \left[A(\omega_r)\cos(\omega_r t) - B(\omega_r)\sin(\omega_r t) \right] \Delta\omega \tag{32}$$

where

$$\omega_r = r\Delta\omega \tag{33}$$

In equations (33) $A(\omega_r)$ and $B(\omega_r)$ are the real and imaginary parts of function $x(\omega)$, respectively.
Secondly, the convolution integral that is expressed in the form:

$$y(t) = \int_0^t f(\xi) x(t-\xi) d\xi \tag{34}$$

is used to compute time response of the system to the forces mentioned before. In equation (34), $f(\xi)$ is the force defined in the form of matrix, and the whole relation can be computed numerically (for example, by using MATLAB software function 'conv' [9]).
Figs. 13-15 show the dynamic response of u_1, u_3, and p_0 respectively. It is seen that the first part of the spectrum has zero quantity, as the force is equal to zero (no force). However, the second part shows vibration, in which the amount is two times bigger in response to limited step contrasting with limited ramp, as it is expected. Also the third part shows vibration after finishing force, and the altitude of vibration is related to the situation that force comes to an end.

4. Conclusions

This paper shows how the DLMT can be used to analyze a complex vibrating system for investigating natural frequencies, time and frequency responses.
The frequencies computed by DLMT are compared with FEM results, in two cases, and as it is shown in Tables 2 and 3 the two methods are differed less than two percent, which confirms the DLMT results. Since the equations of motion are solved exactly in DLMT, the achieved results are in high accuracy.
It is also shown that DLMT can be used to compute the frequency and time responses to different forces, and the model can be set to include the force effects on the system efficiently.
The DLMT can be used for other kind of vibrations, say, torsional and transverse vibrations. Also the present method is straightforward and general and can readily be used in developing a more advanced theory such as vibration of Timoshenko's beams.

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Figures and Tables

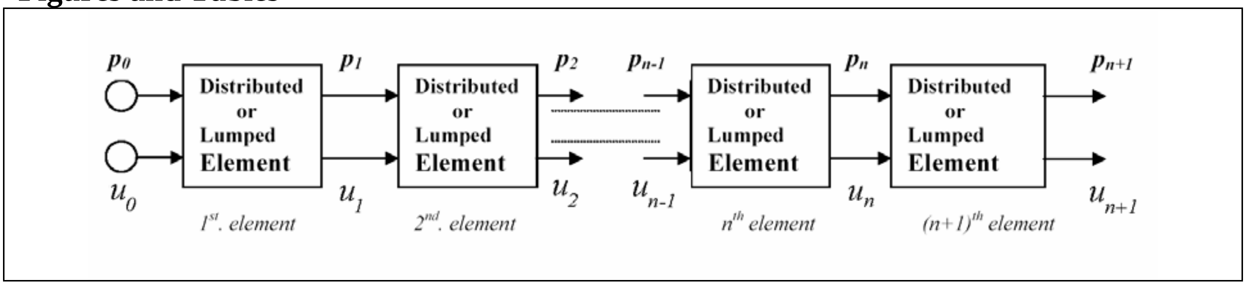

Fig. 1 - General series representation of a distributed lumped parameter system (Hybrid Model)

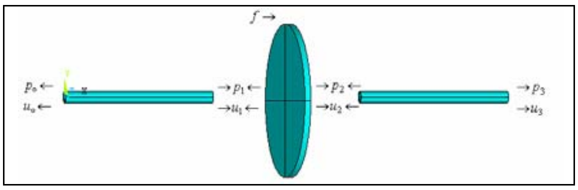

Fig. 2 – General model of rotating rotor system

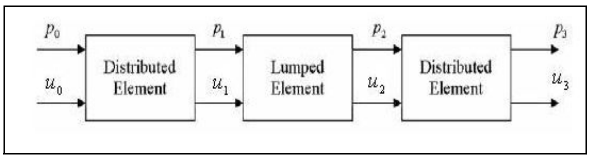

Fig. 3 – Hybrid model of rotating rotor system

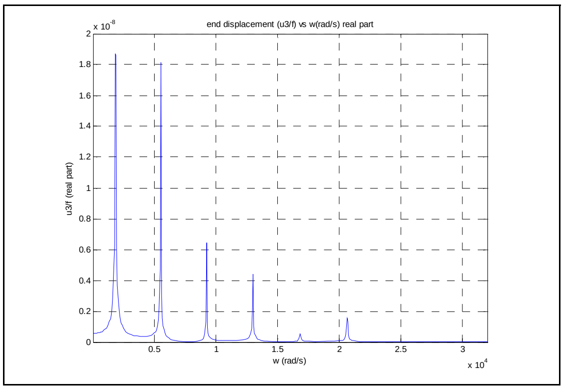

Fig. 4 – Frequency spectrum for clamped-free B.C.'s (u_3 vs. $\omega(rad/s)$)

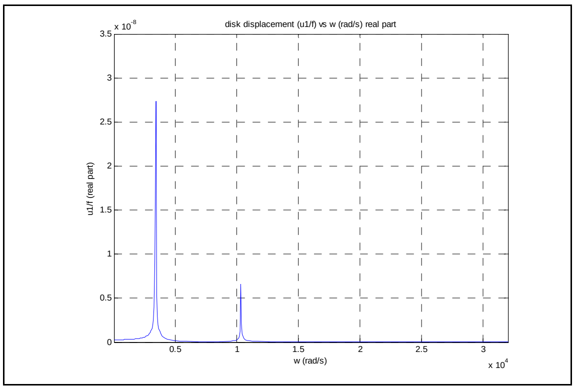

Fig. 5 – Frequency spectrum for clamped-clamped B.C.'s (u_1 vs. $\omega(rad/s)$)

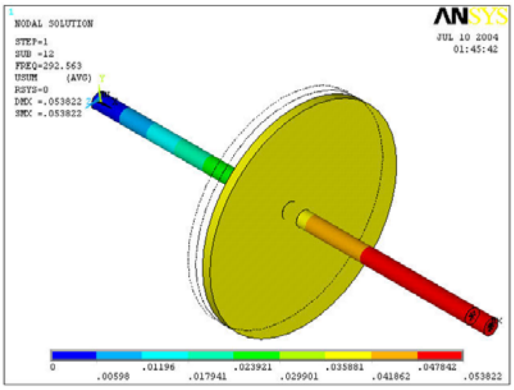

Fig. 6 – 1st mode shape for clamped-free B. C.'s

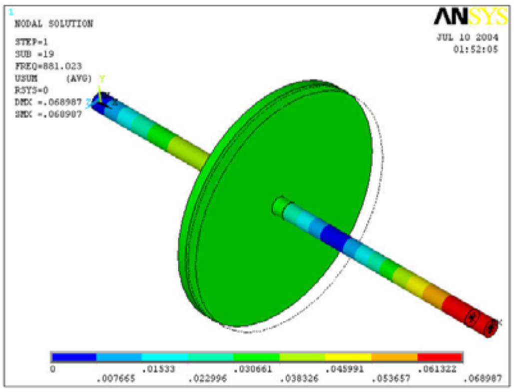

Fig. 7 – 2nd mode shape for clamped-free B. C.'s

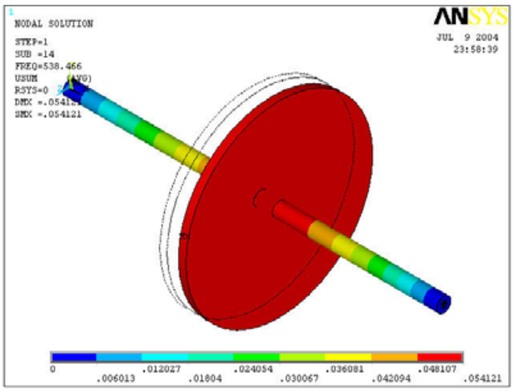

Fig. 8–1st mode shape for clamped-clamped B.C.'s

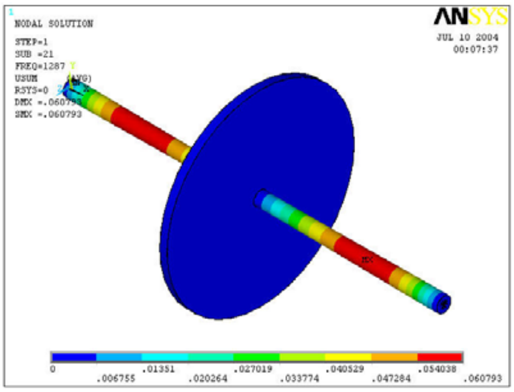

Fig. 9–2nd mode shape for clamped-clamped B.C.'s

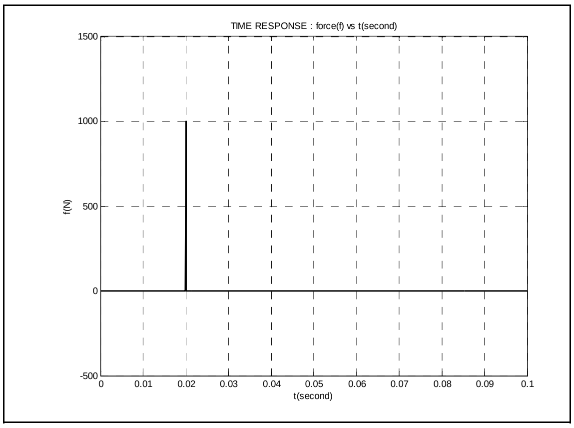

Fig. 10 - Delta force function

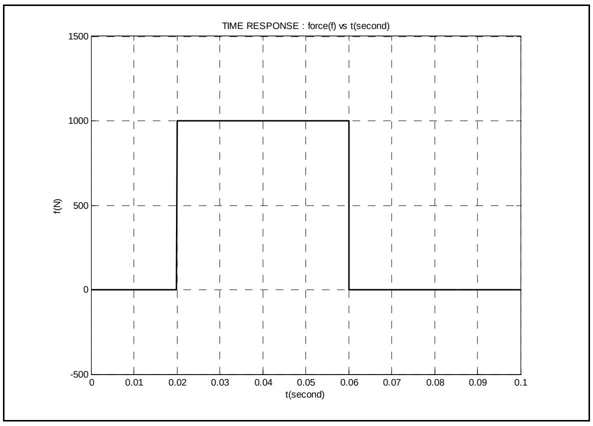

Fig. 11 - Step force function

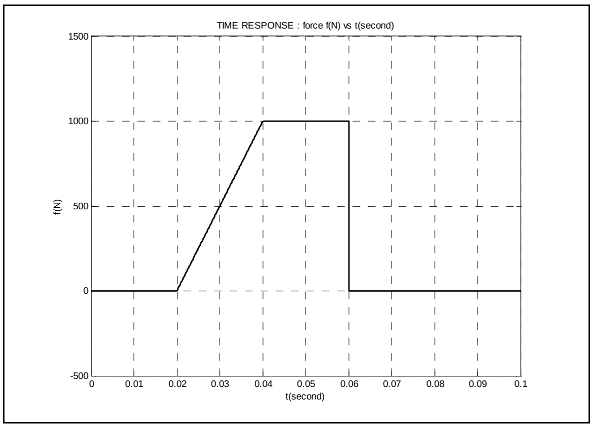

Fig. 12 – Ramp-step force function

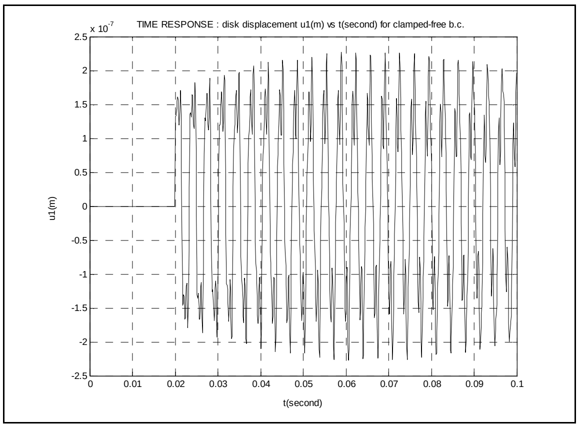

Fig. 13 - Time response of disk displacement under delta force function

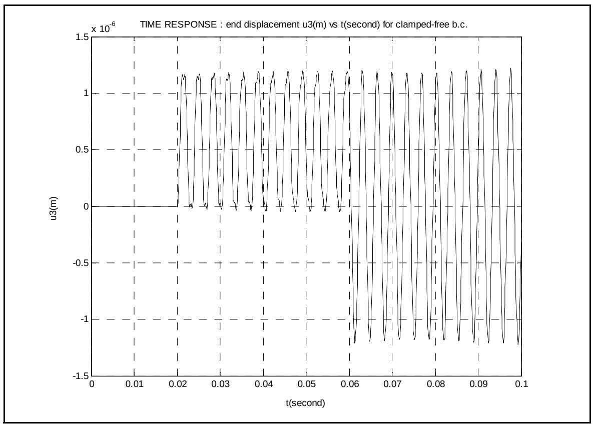

Fig. 14 –Time response of end point displacement of shaft under step force function

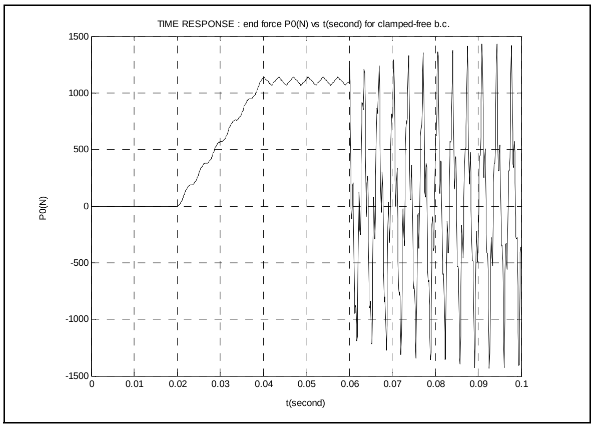

Fig. 15 –Time response of force at the clamped point of shaft under ramp-step force function

Table 1 – Specifications of the system

Shaft Length *2l*	4 m
Shaft Diameter d_{shaft}	0.15 m
Mass of Shaft per Unit Length *m*	137.837 kg
Density of Shaft Material ρ	7800 $\mathrm{kg/m^3}$
Modulus of Elasticity of Shaft *E*	200 GPa
Shear Modulus of Shaft *G*	80 GPa
Mass of Disk	100 kg
Radius of Disk	1 m
Moment of Inertia of Disk	50 $\mathrm{kg.m^2}$
Disk Thickness	0.08 m

Table 2 – Natural frequencies of clamped-free rotating system

Frequency (Hz)	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
DLMT Solution	**290**	**875**	**1464**	**2064**	**2671**	**3285**	**3902**	**4523**
FEM Solution	**292**	**881**	**1480**	**2085**	**2704**	**3321**	**3953**	**4572**
Error Percent (FEM in respect to DLMT)	**0.690**	**0.686**	**1.093**	**1.017**	**1.245**	**1.096**	**1.307**	**1.083**

Table 3 – Natural frequencies of clamped-clamped rotating system

Frequency (Hz)	f_1	f_2	f_3	f_4	f_5	f_6	f_7
DLMT Solution	**537**	**1266**	**1642**	**2532**	**2801**	**3798**	**4001**
FEM Solution	**538**	**1287**	**1651**	**2574**	**2826**	**3860**	**4042**
Error Percent (FEM in respect to DLMT)	**0.186**	**1.659**	**0.548**	**1.659**	**0.893**	**1.632**	**1.025**