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Accelerated Life Testing Based on Proportional Mean Residual Life Model for Multiple Failure Modes

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Abstract: In empirical engineering projects on reliability testing, replacement strategy, spares for repair and reliability growth, the Mean Residual Life (MRL) function may be more relevant than the hazard function or failure time distribution function, although they are mathematically equivalent. However, for modeling purposes, MRL and the hazard functions play somewhat different roles. The MRL summarizes the entire residual life distribution of time t and the hazard function only relates to the risk of immediate failure at time t. When the main concern is the risk of immediate failure, the Proportional Hazards (PH) model has proved to be an extremely useful model. Yet, it may not be the most appropriate model when one is concerned with the remaining lifetime of an individual part at time t as in replacement and repair models. In this study, a new accelerated life testing model based on the Proportional MRL (PMRL) model in the presence of more than one failure mode is presented and its applicability in empirical reliability engineering is shown. The model can provide accurate reliability estimates for multiple failure modes problems and is a useful alternative to the Accelerated Failure Time (AFT) and the PH models. In some cases, our model is even better than the PH model.

Key words: Reliability, accelerated life test, proportional mean residual life, multiple failure modes, maximum likelihood estimation, sum of squared errors

INTRODUCTION

In Accelerated Life Tests (ALT), products are exposed to higher stress conditions (e.g., higher voltage, pressure, or temperature) to produce failures earlier than at typical conditions.

Acceleration in lab tests is justified because products are expected to survive a considerable length of time (e.g., months, years, decades) under typical operating conditions. A model is then fit to the data collected and the results are used to estimate quantities of interest, e.g., quantiles and hazard rates at use conditions through extrapolation. The first important step in accelerated life testing is to determine a test plan given constraints such as maximum test duration and test unit availability. In general, this involves specifying at what levels to test (e.g. temperature settings) and how many units to test at each level.

The inference procedures (or models) are classified into three types: statistics based models. Physics-statistics based models and physics-experimental based models (Pham, 2003). Furthermore, the statistics-based models are classified into two categories parametric and non-parametric models. Parametric models that are based on AFT assumption, assume that the failure time data follow a distribution such as exponential or Weibull. It is also assumed that the failure times at different stress levels are linearly related to each other. Moreover, the failure time distribution at stress level s_1 is expected to be the same at different stress levels s_2,s_3,... as well as under normal operating conditions. In other words, the shape parameters of the distributions are the same for all stress levels (including normal conditions), but the scale parameters may be different. Thus, the relationship between the operating conditions and stress conditions is:

$$\lambda(t:z) = \lambda_0(e^{\beta z}t)e^{\beta z} \quad (1)$$

where, $\lambda(t,z)$ is the hazard function at time t and stress vector Z, $\lambda_0(t)$ is the baseline hazard rate function; and β is the coefficient of stress covariate Z (Sarhan, 2007).

In the non-parametric models there is no assumption of the failure time distribution, i.e. no predetermined failure time distribution is required. Cox (1972) proportional hazards (PH) model is the most widely used among the non-parametric models. It is expressed as:

$$\lambda(t;z) = \lambda_0(t)\exp[\beta z] = \lambda_0(t)\exp\left(\sum_{j=1}^{p}\beta_j z_j\right) \quad (2)$$

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where, $z = (z_1, z_2, Y, z_p)'$ is a column vector of the covariates (or applied stresses), $\beta = (\beta_1 \beta_2 \ldots \beta_p)$ is a column vector of the unknown coefficients and $\lambda_0(t)$ is the baseline hazard rate function.

The PH model implies that $\lambda(t;\ z_1)$ is directly proportional to $\lambda(t;\ z_2)$. This is the so-called PH model's hazard rate proportionality assumption.

If the proportionality assumption is violated and there are one or more covariates that totally occur on q levels, a simple extension of the PH model is stratification (Kalbfleisch and Prentice, 2002), as given below:

$$\lambda_j(t;z) = \lambda_{oj}(t)\exp[\beta z], \quad j = 1, \ldots, q \tag{3}$$

A partial likelihood function can be obtained for each of the q strata and β is estimated by maximizing the multiplication of the partial likelihood of the q strata. The baseline hazard rate λ_{0j} (t), estimated as before, is completely unrelated among the strata. This model is most useful when the covariate is categorical and of no direct interest.

Another extension of the PH model includes the use of time-dependent covariates. Kalbfleisch and Prentice (2002) classified the time-dependent covariates as internal and external. An internal covariate is the output of a stochastic process generated by the unit under study and can be observed as long as the unit survives and is uncensored. An external covariate has a fixed value or is defined in advance for each unit under study.

Many other extensions exist in the literature (Pham, 2003). However, one of the most generalized extensions is the extended linear hazard regression model proposed by Elsayed *et al.* (2006). Both accelerated failure time model and PH model are indeed special cases of the generalized model, whose hazard function is:

$$\lambda(t;Z) = \lambda_o \left\{ t e^{(\beta_0 + \beta_1 t)z} \right\} e^{(\alpha_0 + \alpha_1 t)z} \tag{4}$$

where, $\lambda(t;\ z)$ is the hazard rate at time t and stress vector z; $\lambda_0(\$)$ i s the baseline hazard function; and β_0, β_1, α_0, α_1 are constants. This model has been validated through extensive experimentation and simulation testing (Pham, 2003).

All the above models are based on failure-rates proportionality and failure-times proportionality. Oakes and Dasu (1990) originally proposed the concept of the proportional mean residual life (PMRL) by analogy with Cox (1972) PH model. The concept of MRL is based on conditional expectations and has been of much interest in actuarial science, survival studies and reliability theory. In the last two decades, however, reliability engineers, statisticians and others have shown increasing interest in the MRL and derived many useful results.

Two distributions with reliability functions R_0 and R are said to have proportional MRL functions if, for some $\theta > 0$, $e(t) = \theta e_0(t)$, $\forall t$ where, e(t) is the mean residual life under accelerated conditions, θ is a constant and $e_0(t)$ is the mean residual life under normal conditions.

In reliability studies of repair and replacement strategies, the MRL function may be more relevant than the hazard function or failure time distribution function, although they are mathematically equivalent, that is by knowing one, the other can be determined:

$$\lambda(t) = 1 + e'(t)/e(t) \tag{5}$$

However, for modeling purposes, MRL and the hazard function play slightly different roles. The former summarizes the entire residual life distribution of time t and the latter only the risk of immediate failure at time t. It is possible for the MRL function to exist, but for the failure rate function not to exist (e.g., the standard Cantor ternary function), although sometimes it is possible for the failure rate function to exist but the MRL function not to exist. For example,

$$f(t) = 2/\pi(1+t^2) \quad \text{for} \quad t \geq 0 \tag{6}$$

When, the main concern is the risk of immediate failure, the proportional hazards model has proved to be extremely useful. However, it may not be the most appropriate model when one is concerned with the remaining lifetime of an individual at time t. In this case, when the ALT model is developed based on the PMRL, it will provide a useful alternative to the standard models for ALT, which includes the accelerated failure model and the proportional hazards.

Let $X \geq 0$ be a continuous random variable with reliability function R(x) and finite expectation μ. The MRL is defined to be:

$$e(x) = E(X - x \mid X > x) = \frac{\int_x^\infty R(u)du}{R(x)} = \frac{\int_x^\infty u \cdot f(u)du}{R(x)} - x \tag{7}$$

Gupta and Kirmani (1998) presented details of the implications of the PMRL and Zhao and Elsayed (2005a, b) modeled an accelerated life test based on mean residual life.

Chen *et al.* (2005) presented a semiparametric estimation of the Proportional Mean Residual Life model in the presence of censoring. In this study, an approach to ALT planning when product failure is caused by two or more failure modes is presented.

The problem of multiple failure modes encompasses the study of any failure process in which there is more than one distinct cause or type of failure (Balasooriya and Low, 2004; Zhang and Elsayed, 2006; Kundu and Sarhan, 2006). This is very important and should be considered in reliability studies. The problem may be described as follows:

For a given unit, let T_i be a random variable with cumulative distribution function $F_i(t)$, $i = \{1,2,..., k\}$, where k is the number of failure modes or types. It is assumed that the T_i 's are not observable and only the time of failure, $T \geq 0$, $T = \min(T_1, T_2 \ldots, T_k)$ and the cause of the failure, j, among a finite set of possible causes, say $j \in \{1,2, \ldots, k\}$, which may be censored may be observed. A regression vector z may also be available to record characteristics of the under study subject. Some components of z may be time dependent, that is $z = z(t)$. For the accelerated life testing problem with multiple failure modes, a set of covariates z can also be observed to reflect the characteristics of the tested components, for example, the environmental conditions that the components experienced in the test.

However, existing reliability methods for multiple failure mode problems deal only with products operating at normal conditions subject to hard (catastrophic) failures, which imply abrupt and complete cessation of the product's function.

NONPARAMETRIC APPROACHES WITH MULTIPLE FAILURE MODES PROBLEM

The analysis of failure data with applied stresses at accelerated conditions often involves complex and not well-known shapes of time to failure distributions. To avoid making additional assumptions that would be difficult to test, nonparametric regression models appear to be more attractive than the parametric ones which assume a specific distribution. Further contributions on the subject are given in Kalbfleisch and Prentice (2002) and Park (2005).

Consider inference on the relationship between cause-specific hazard functions and regression vectors or function z. For example, proportional hazards modeling in which the cause-specific hazard function at time t depends on z only in terms of the concurrent value z(t) is:

$$\lambda_j(t;Z) = \lambda_{oj} \exp[\beta_j z(t)], \quad j = 1,...,m \tag{8}$$

Both the shape functions λ_{0j} and regression coefficients β_j have been permitted to vary arbitrarily over the m failure types.

Let $t_{ji} < \ldots < t_{ikj}$ denote the time of k_j failures of type j, j = 1, Y, m and let Z_{ji} be the regression function for the individual that fails at t_{ji}. The method of partial likelihood then gives:

$$L(\beta_1,...,\beta_m) = \prod_{j=1}^{m} \prod_{i=1}^{k_j} \left(\frac{\exp[\beta_j z_{ji}(t)]}{\sum_{l \in R(t_{ji})} \exp[\beta_j z_l(t_{ji})]} \right) \tag{9}$$

Estimation and comparison of the β_j's can be conducted by applying standard asymptotic likelihood techniques individually to the m factors. The functions $R_j(t;\ z)$ can be estimated at specified Z upon inserting the maximum likelihood estimators from the above Equation. The corresponding estimators of the cumulative incidence can be obtained simply by inserting the appropriate estimators for R and λ_i functions.

$$I_j(t;z) = P(T < t, J = j; Z) = \int_0^t \lambda_j(u;z) R(u;z) du, \quad j = 1,...,m \tag{10}$$

The cause-specific hazard functions could similarly be modeled using an accelerated failure time model:

$$\lambda_j(t;Z) = \lambda_{oj} \left\{ t e^{\beta_j z(t)} \right\} e^{\beta_j z(t)}, \quad j = 1,...,m \tag{11}$$

It would be necessary to restrict the covariate to be fixed or a step function in order to preserve the multiplicative relationship between covariates and failure time.

PMRL MODEL AND STATISTICAL INFERENCE FOR MULTIPLE FAILURE MODES

Two distributions with reliability functions R_0 and R and with mean residual lives at time x of e(x) and $e_0(x)$, respectively, are said to have proportional MRL functions, if they are related as follows:

$$e(x) = \theta e_0(x), \quad \forall x, \theta > 0 \tag{12}$$

therefore:

$$R(x,\theta)=\frac{e(0)}{e(x)}\exp\left(-\int_x^{\infty}\frac{du}{e(u)}\right)=R_0(x)\left(\frac{\int_x^{\infty}R_0(u)du}{\mu_0}\right)^{\frac{1}{\theta}-1} \quad (13)$$

Where:

$$\mu_0=e_0(0)$$

It is assumed that there are more than one independent failure modes. That is, time to failure is viewed as the minimum of p latent failure times. Here only two failure modes are considered. The procedure is similar for the cases with p failure modes.

Consider N components subjected to accelerated life testing with an applied stress z. The test is conducted until all components fail, i.e., no censoring. A component may fail in two failure modes: failure in mode 1 and failure in mode 2.

The collected data from the test units with two failure modes have three features:

- Failure mode δ_1, δ_2
- Failure times T_1, T_2 respectively

$$U=\min(T_1,T_2) \quad (14)$$

(U,δ) are observable.
T_1 and T_2 are independent

- Stress level(Z)

then

$$R_u(t)=P(U>t)=P(\min\{T_1,T_2\}>t)=P(T_1>t,T_2>t)=P(T_1>t)P(T_2>t)=R_1(t)R_2(t) \quad (15)$$

and

$$\lambda(t)=\lambda_1(t)+\lambda_2(t)$$
$$f(t)=\lambda(t)R(t),\quad f_1(t)=\lambda_1(t)R_1(t),$$
$$f_2(t)=\lambda_2(t)R_2(t), \quad (16)$$

but

$$f(t)\neq f_1(t)+f_2(t)$$

From the assumptions, the likelihood function can be represented as:

$$L=\prod_{i=1}^{n}f(t_i)=\prod_{i=1}^{n}(\lambda_1(t_i)+\lambda_2(t_i))\times\exp\left[-\int_0^i(\lambda_1(u)+\lambda_2(u))du\right] \quad (17)$$

and

$$\ln L=\sum_1^n\ln(\lambda_1(t_i)+\lambda_2(t_i))-\sum_1^n\int_0^i(\lambda_1(u)+\lambda_2(u))du \quad (18)$$

where, lnL is a function of $\lambda_1(\cdot)$, $\lambda_2(\cdot)$. We can maximize lnL to estimate $\lambda_1(\cdot)$, $\lambda_2(\cdot)$. Suppose $\lambda_1(\cdot)$, $\lambda_2(\cdot)$ can be represented by the PMRL model separately:

$$\lambda_1(t;z)=\frac{e'_{10}(t)+\exp(-\beta_1^Tz)}{e_{10}(t)}$$
$$\lambda_2(t;z)=\frac{e'_{20}(t)+\exp(-\beta_2^Tz)}{e_{20}(t)}$$
$$\lambda\ (t;z)=\frac{e'_{10}(t)+\exp(-\beta_1^Tz)}{e_{10}(t)}+\frac{e'_{20}(t)+\exp(-\beta_2^Tz)}{e_{20}(t)} \quad (19)$$

and

$$R_j(t;z)=\frac{e_{j0}(0)}{e_{j0}(t)}\exp\left(-\int_0^i\frac{du}{\exp(\beta_jz)e_{j0}(u)}\right)$$
$$R\ (t;z)=\frac{\prod e_{j0}(0)}{\prod e_{j0}(t)}\exp\left(-\sum\left(\int_0^i\frac{du}{\exp(\beta_jz)e_{j0}(u)}\right)\right)=\frac{\prod e_{j0}(0)}{\prod e_{j0}(t)}\exp(-E_i) \quad (20)$$

Where:

$$E_i=\int_0^i\frac{du}{\exp(\beta_jz)e_{j0}(u)} \quad (21)$$

$$f(t;z)=\sum_{j=1}^{2}\lambda_j(t;z)\cdot R\ (t;z) \quad (22)$$

Substitute the specific form of e_{10}, e_{20} into the log likelihood function and consider censoring, then we have:

$$L(\beta)=\prod_{u} f(t_u,z_u,\beta)\prod_{c} R(t_c,z_c,\beta)= \\ \sum_{u} \ln f(t_u,z_u,\beta)+\sum_{c} \ln R(t_c,z_c,\beta)= \\ \sum_{c} \ln(\sum_{j=1}^{2}\lambda_j(t_u;z_u)\cdot R(t_u;z_u))\sum_{c} \ln R(t_c;z_c) \quad (23)$$

Let's only consider the scenario that there is no censoring, thus the log likelihood function is as follows:

$$\ln L = \\ \sum_{i}^{n} \ln\{[\lambda_1(t_i;z_i)+\lambda_2(t_i;z_i)]R_1(t_i;z_i)\}= \\ \sum_{i}^{n} \ln\left\{\left[\frac{e'_{10}(t_i)+\exp(-\beta_1^T z_i)}{e_{10}(t_i)}+ \right.\right. \\ \left.\frac{e'_{20}(t_i)+\exp(-\beta_2^T z_i)}{e_{20}(t_i)}\right]R_1(t_i;z_i)\Big\}= \\ \sum_{i}^{n} \ln\left\{\left[\frac{e'_{10}(t_i)+\exp(-\beta_1^T z_i)}{e_{10}(t_i)}+ \right.\right. \\ \left.\frac{e'_{20}(t_i)+\exp(-\beta_2^T z_i)}{e_{20}(t_i)}\right]\times\frac{\prod_{j=1}^{2} e_{j0}(0)}{\prod_{j=1}^{2} e_{j0}(t)}\times \\ \exp(-\sum_{j=1}^{2} E_i)\Big\} \quad (24)$$

SIMULATION RESULTS

MATLAB was used for computer simulations to demonstrate the use of the proportional mean residual life model in modeling the failure times obtained from accelerated life testing with multiple failure modes in order to show its applicability in the reliability field.

The exponential function was used as the baseline function of the PMRL model in which there are 6 unknown parameters, a_1, a_2, b_1, b_2, β_1, β_2 and the Baseline exponential Function is:

$$e_{i0}(t)=\exp(a_i+b_i t) \quad (25)$$

substituting (25) in the PMRL model, we obtain:

$$e_i(t;z)=\exp(a_i+b_i t)\exp(\beta_i^T z) \quad (26)$$

In order to perform the simulation, four groups of weibull distribution data with two failure modes and two stress levels were generated. Then failure times for

Table 1: Estimated parameters of the PMRL model

Parameters	Exponential optimal result
a_1	0.61352715321E+01
b_1	-0.98322141 61E-04
a_2	0.69881089908E+01
b_2	0.11348666712E-03
β_1	-0.3662834323E-02
β_2	-0.137964537374E-02

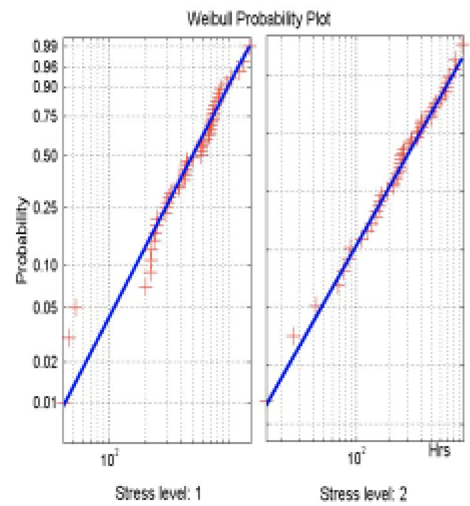

Fig. 1: Generated data

two stress levels were extracted as shown in Fig. 1. The MATLAB optimization toolbox was used for maximizing lnL. Maximizing lnL is equivalent to minimizing (lnL). The selection of the initial value is very important. For the exponential baseline mean residual life case, the generated data of failure modes 1 and 2 was fitted with the Weibull model separately, then let the initial value of the parameter a_1 and a_2 in our model,

$$a_i=\ln(\text{MTTF}_i) \quad (27)$$

β_1, β_2 equal to zero in present model.

After generating the failure time data, the PH, the Kaplan-Meier and the PMRL models were fitted to the data. The estimated parameters are shown in Table 1 based on which this model can be used to estimate the reliability at a specific stress level. Figure 2 shows the reliability estimation at stress level 2 for the PMRL model, PH model and the true model.

Figure 3 shows the probability distribution function (pdf) of the true model and the PMRL model.

In Fig. 4, the PMRL model and the PH models are compared with the Kapaln-Meier model and Fig. 5 shows all models at stress level 1.

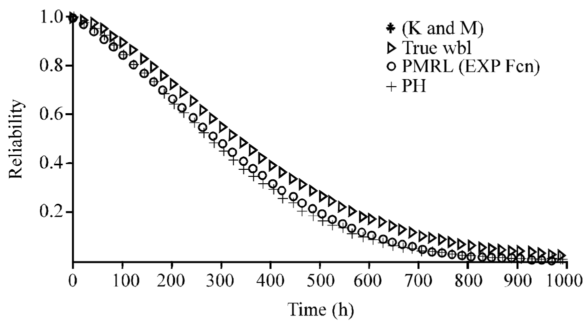

Fig. 2: Reliability estimation at stress level 2 for the PMRL Model, the PH Model and the true Model

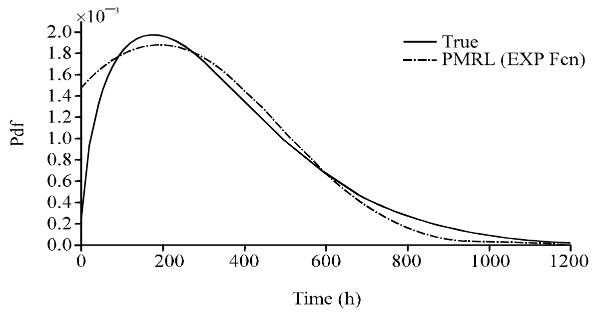

Fig. 3: Pdf of the true and PMRL model

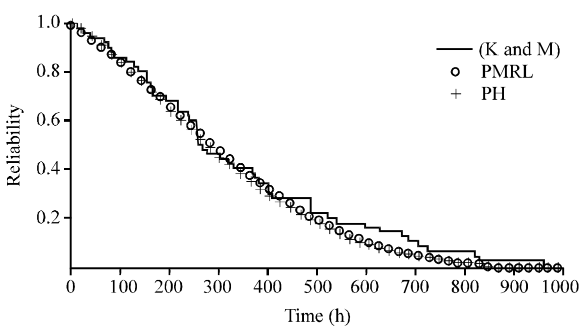

Fig. 4: Reliability estimation at stress level 2 for the PMRL model, the PH model and the K and M model

Figure 5 shows that the PMRL model has better results in modeling the data obtained from accelerated test with multiple failure modes.

Table 2 shows the sum of squared error (SSE) between PMRL estimates, true data and results obtained from K and M model at stress level 1 and Table 3 shows the same statistical measures at stress level 2. Based on the results shown in Table 2 and 3, the PMRL model gives very promising results, where the sum of squared error between PH estimates and the observed data is 0.7141 and the same statistical measure for PMRL model is only 0.5037 at stress level 1 and the sum of squared

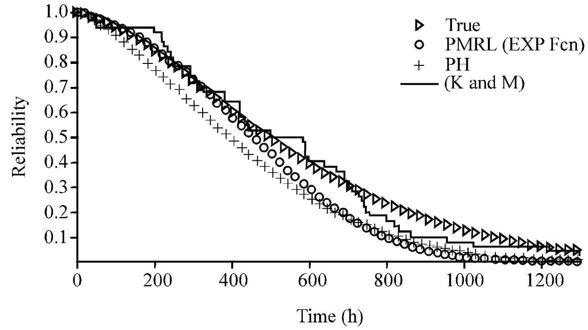

Fig. 5: Reliability estimation at stress level 1 for the PMRL model, the PH model, the true model and the K and M model

Table 2: SSE for models at stress 1

Models	SSE PMRL	SSE PH
True	0.5037	0.7241
K and M	0.2744	0.6432

Table 3: SSE for models at stress 2

Models	SSE PMRL	SSE PH
True	0.1925	0.3036
K and M	0.0439	0.0590

Table 4: Goodness of fit for model

	PMRL	PH
Goodness of fit	64%	36%

errors at stress level 2 for the PH model and PMRL are 0.3036 and 0.1925. Therefore, the PMRL model is a useful alternative model. In some cases, it is even better than the PH model.

MODEL VERIFICATION

In order to compare the proposed model with the PH model, 120 data sets distributed as Weibull were generated each of which included four groups of Weibull distribution data with different stress levels. After generating the failure data, the PH model and the PMRL model were fitted to the data and the sum of squared error between model estimates and the observed data was calculated, as shown in Table 4.

Based on this Table 4 are 77 data sets in which the PMRL model provides a better estimate than the PH model. These results show that the PMRL model provides a useful alternative to the PH model and gives very promising results.

Similar with the PH model, the PMRL model implies that the ratio of the MRL functions for any two units associated with different vectors of covariates, z_1 and z_2, is constant with respect to time. This means that $e(t; z_1)$ is directly proportional to $e(t; z_2)$. The PMRL model is a valid model to analyze ALT data only when the data satisfy its proportional mean residual life assumption. Therefore, it is very important to check the validity of the PMRL model and the assumption of the covariates' multiplicative effect before applying it to the failure data. Weibull distribution was used in this research to generate data to satisfy this assumption.

CONCLUSIONS

In this study, a new accelerated life testing based on Proportional Mean Residual Life model in the presence of more than one failure mode is presented and its applicability in reliability is shown. The model utilizes the data at accelerated conditions to estimate the reliability measures at normal operating conditions.

Moreover, the use of the PMRL that is modeled is shown in modeling ALT's with multiple failure modes, where exponential function is used to present the baseline mean residual function. The average square error between the PMRL model based on this baseline function and the true data is very small. This means that the model can provide accurate reliability estimates for multiple failure mode problems and is a useful alternative to the accelerated failure time (AFT) and the proportional hazards (PH) models. In some cases, this model is even better than the PH model.

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