FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, we discuss about various definitions of fuzzy topological spaces. Some obstacles which appear in theory of fuzzy topological spaces will be discussed.

1. introduction

The concept of a fuzzy set, which was introduced by Zadeh in [4] motivated some authors to generalizing many concepts of general topology to fuzzy topological space. In this paper, we investigate some of these results to find more appropriate definition of a fuzzy topology on a space, which gives generalization of basic concepts of topology such as open sets, neighborhoods, continuity and compactness.

2. Fuzzy topology

We begin with several preliminary definitions.

Let X be a set, by a fuzzy set in X we mean a function $\mu : X \to [0,1]$. In fact, $\mu(x)$ represents the degree of membership of x in the fuzzy set X. For example, every characteristic function of a set, is a fuzzy set. The characteristic functions of subsets of a set X are referred to as the crisp fuzzy sets in X.

Let A and B be fuzzy sets in a space X, with the grades of membership of x in A and B denoted by $\mu_A(x)$ and $\mu_B(x)$, respectively. Then we say that

(1) $A = B$ if and only if $\mu_A \equiv \mu_B$.

(2) $A \subset B$ if and only if $\mu_A \leq \mu_B$.

(3) $C = A \cup B$ if and only if $\mu_C \equiv \max\{\mu_A, \mu_B\}$.

(4) $D = A \cap B$ if and only if $\mu_D \equiv \min\{\mu_A, \mu_B\}$.

(5) $E = A'$ if and only if $\mu_E \equiv 1 - \mu_A$.

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More generally, for a family of fuzzy sets $\mathcal{A} = \{A_\alpha : \alpha \in I\}$, the union $C = \bigcup_{\alpha\in I} A_\alpha$, and the intersection $D = \bigcap_{\alpha\in I} A_\alpha$ are respectively defined by

$$\mu_C(x) = \sup\{\mu_{A_\alpha}(x) : \alpha \in I\} \quad x \in X$$

and

$$\mu_D(x) = \inf\{\mu_{A_\alpha}(x) : \alpha \in I\} \quad x \in X.$$

The symbol $\emptyset$ will be used to define the empty fuzzy set $\mu_\emptyset(x) = 0$ for all $x \in X$. In [1], Chang defined the following notion of fuzzy topological space:

Definition 2.1. A fuzzy topology is a family $\mathcal{T}$ of fuzzy sets in X which satisfies the following conditions:
(a) $\emptyset, X \in \mathcal{T}$,
(b) If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$,
(c) If $\{A_\alpha : \alpha \in I\}$ is a family in $\mathcal{T}$, then $\bigcup_{\alpha\in I} A_\alpha \in \mathcal{T}$.

$\mathcal{T}$ is called a fuzzy topology for X and the pair $(X, \mathcal{T})$ is a fuzzy topological space. Every element of $\mathcal{T}$ is called a $\mathcal{T}$-open fuzzy set. A fuzzy set is $\mathcal{T}$-closed if its complement is $\mathcal{T}$-open.

Example 2.2. (a) Let (X, τ) be a topological space (in the ordinary sense). If elements of τ are identified with their characteristic functions, then (X, τ) is a fuzzy topological space.
(b) The collection of all crisp fuzzy sets in X is a fuzzy topology on X.
(c) The intersection of any family of fuzzy topologies on a set X is a fuzzy topology on X.

Let X and Y be sets and $f : X \to Y$ be a function. Let B be a fuzzy set in Y with membership function μ_B. Then the inverse of B, written as $f^{-1}(B)$ is a fuzzy set in X whose membership function is defined by

$$\mu_{f^{-1}(B)}(x) = \mu_B(f(x)) \quad \forall x \in X.$$

Conversely, let A be a fuzzy set in X with membership function $\mu_A(x)$. The image of A, written as $f(A)$, is a fuzzy set in Y whose membership function is given by

$$\begin{aligned} \mu_{f(A)}(y) &= \sup\{\mu_A(x) : x \in f^{-1}(y)\} \quad \text{if } f^{-1}(y) \text{ is not empty} \\ &= 0 \text{ otherwise,} \end{aligned}$$

for all $y \in Y$, where $f^{-1}(y) = \{x : f(x) = y\}$.
The usual properties relating images and inverse image of subsets to unions, intersections and complements also hold for fuzzy sets (see [1] Theorem 4.1).

Definition 2.3. A function f from a fuzzy topological space $(X, \mathcal{T}_X)$ to a fuzzy topological space $(Y, \mathcal{T}_Y)$ is fuzzy continuous if the inverse image of every $\mathcal{T}_Y$-open fuzzy set is $\mathcal{T}_X$ open.

Clearly if f is a fuzzy continuous function on X to Y and g is a fuzzy continuous function on Y to Z, the the composition gof is a fuzzy continuous function on X to Z. We refer the reader to [1] for elementary properties of fuzzy continuous mappings.
Now, we give the notion of fuzzy compact space, which is given by Chang in [1].

Definition 2.4. A family $\mathcal{A}$ of fuzzy sets is a cover of a fuzzy set B if $B \subset \bigcup\{A : A \in \mathcal{A}\}$. $\mathcal{A}$ is said to be an open cover for B if each member of $\mathcal{A}$ is an open fuzzy set. A subcover of $\mathcal{A}$ is a subfamily of $\mathcal{A}$ which is also a cover. A fuzzy topological space $(X, \mathcal{T})$ is compact if each open cover of X has a finite subcover.

Definition 2.5. A family $\mathcal{A}$ of fuzzy sets has the finite intersection property if the intersection of the members of each finite subfamily of $\mathcal{A}$ is nonempty.

Theorem 2.6. [1] *A fuzzy topological space is compact if and only if each family of closed fuzzy sets which has the finite intersection property has a nonempty intersection.*

Theorem 2.7. [1] *Let f be a fuzzy continuous mapping from $(X, \mathcal{T}_X)$ onto $(Y, \mathcal{T}_Y)$. If $(X, \mathcal{T}_X)$ is compact, then so is $(Y, \mathcal{T}_Y)$.*

Definition 2.8. Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in I}$ be a family of fuzzy topological space. The product topology $\mathcal{T} = \Pi_{\alpha \in I} \mathcal{T}_\alpha$ on the set $X = \Pi_{\alpha \in I} X_\alpha$ is the coarsest fuzzy topology on X making all projection mappings $\pi_\alpha : X \to X_\alpha$ fuzzy continuous.

The following example shows that the product of countably many compact fuzzy topological spaces may be non-compact.

Example 2.9. Let $X_n = \mathbb{R}$ for $n = 1, 2, \ldots$. Let μ_m be the constant fuzzy set $\frac{m-1}{m}$ and $\mathcal{T}_n = \{\emptyset, X_n, \mu_m.\chi_{[-n,n]} : m = 1, 2, \ldots\}$. Then $(X_n, \mathcal{T}_n)$ is compact for each n, since every fuzzy open cover of X_n contains X_n. Let $(X, \mathcal{T}) = (\prod_{n \in \mathbf{N}} X_n, \prod_{n \in \mathbb{N}} \mathcal{T}_n)$. For each $(m, n) \in \mathbb{N} \times \mathbb{N}$ Let $V_{(m,n)}$ be the fuzzy open set with the membership

$$\pi_m^{-1} \mu_n.\chi_{[-n,n]} = \mu_m.\chi_{[-n,n]} o \pi_m.$$

Then for each $x = \{x_m\} \in X$, there is some $m \in \mathbb{N}$ such that $|x_m| < n$, thus

$$\mu_m.\chi_{[-n,n]} o \pi_m(x) = \mu_m.\chi_{[-n,n]}(x_m) = \begin{cases} \frac{m-1}{m} & |x_m| \leq n \\ 0 & |x_m| > n \end{cases}$$

If $\varepsilon > 0$, there is some $m \in \mathbb{N}$ such that $\frac{1}{m} < \varepsilon$, hence $\mu_m.\chi_{[-n,n]} o \pi_m(x) > 1 - \varepsilon$. It follows that $x \in \bigcup_{(m,n)\in\mathbb{N}\times\mathbb{N}} V_{(m.n)}$. That is $\mathcal{A} = \{V_{(m,n)} : (m,n) \in \mathbb{N}\times\mathbb{N}\}$ covers X. However, any finite subset $\{V_{(m_i,n_i)} : i = 1, \dots, k\}$ of $\mathcal{A}$ does not cover X. Since if $n_0 > \max\{n_1, \dots, n_k\}$ and $x = (n_0, n_0, n_0, \dots)$ then $\mu_{m_i}.\chi_{[-n_i,n_i]} o \pi_{m_i}(x) = 0$ for each $1 \leq i \leq k$. It follows that any finite subset of fuzzy open cover $\mathcal{A}$ does not cover X. Hence $(X, \mathcal{T})$ is not compact.

Corollary 2.10. *Tychonoff Theorem does not hold for fuzzy topological spaces. In fact even countable product of compact fuzzy topological spaces may fail to be compact.*

Another obstacle is that some constant functions from one fuzzy topological space to another fail to be continuous in fact using the definition, we get to the following result.

Proposition 2.11. *Let (X, τ) be a fuzzy topological space. Then every constant function from (X, τ) into another fuzzy topological space is fuzzy continuous if and only if τ contains all constant fuzzy sets in X.*

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