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Characterizations based on Rényi entropy of order statistics and record values

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Abstract

Two different distributions may have equal Rényi entropy; thus a distribution cannot be identified by its Rényi entropy. In this paper, we explore properties of the Rényi entropy of order statistics. Several characterizations are established based on the Rényi entropy of order statistics and record values. These include characterizations of a distribution on the basis of the differences between Rényi entropies of sequences of order statistics and the parent distribution.

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1. Introduction

Suppose that $X_1, \ldots, X_n$ are independent and identically distributed (iid) observations from an absolutely continuous cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. The order statistics of the sample are defined by the arrangement of $X_1, \ldots, X_n$ from the smallest to the largest, denoted as $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{n:n}$. These statistics have been used in a wide range of problems, including robust statistical estimation, detection of outliers, characterization of probability distributions and goodness-of-fit tests, entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials; for more details, see Arnold et al. (1992), David and Nagaraja (2003) and references therein.

Let $X_1, X_2, \ldots$ be a sequence of iid random variables having an absolutely continuous cdf $F(x)$ and pdf $f(x)$. An observation X_j is called an upper record value if its value exceeds that of all previous observations. Thus, X_j is an upper record if $X_j > X_i$ for every $i < j$. Record data arise in a wide variety of practical situations. Examples include industrial stress testing, meteorological analysis, hydrology, seismology, sporting and athletic events, and oil and mining surveys. Properties of record data have been studied extensively in the literature. Interested readers may refer to the books by Arnold et al. (1998) and Nevzorov (2001).

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In reliability theory, order statistics and record values are used for statistical modeling. The $(n-m+1)$th order statistics in a sample of size n represents the life length of an m-out of-n system. Record values are used in shock models and *minimal repair systems* (see Kamps, 1994). Several authors have studied the subject of characterization of F based on the properties of order statistics and record values. The papers of Huang (1975), Nagaraja (1988), Nagaraja and Nevzorov (1997), Stepanov (1994), Balakrishnan and Balasubramanian (1995), Balakrishnan and Stepanov (2004), Abu-Youssef (2003), Park and Zheng (2004), Hofmann et al. (2005) and Raqab and Awad (2000) contain characterizations based on order statistics and record values.

The Shannon entropy of a random variable X is a mathematical measure of information which measures the average reduction of uncertainty of X. The Rényi entropy is a generalization of Shannon entropy and is known to be of importance in cryptography (see Cachin, 1997), resolution in time–frequency (see Knockaert, 2000). Since, for a given α, two different distributions may have the same Rényi entropy, a distribution cannot be determined by its Rényi entropy. We study conditions under which the Rényi entropy of order statistics and record values can uniquely determine the parent distribution F.

The rest of this paper is organized as follows. Section 2 contains some preliminaries. In Section 3, we present some characterizations based on the Rényi entropy of a sequence of order statistics; also we characterize the exponential model based on the difference between Rényi entropy of the first order statistic and the parent distribution. In Section 4, we show that F can be uniquely determined by the equality of Rényi entropy of record values.

2. Preliminaries

The entropy of order α or Rényi entropy of a distribution (Rényi, 1961) is defined as

$$
\begin{aligned}
H_\alpha(X) &= \frac{1}{1-\alpha}\log\int_{-\infty}^{+\infty} f^{\alpha}(x)\,\mathrm{d}x \\
&= \frac{1}{1-\alpha}\log E[f(X)]^{\alpha-1} \\
&= \frac{1}{1-\alpha}\log E_{f_X,\alpha}[r_X^{\alpha-1}(X)] - \frac{\log\alpha}{1-\alpha}, \qquad (1)
\end{aligned}
$$

where $\alpha>0$, $\alpha\neq 1$, and $r_X(t)=f(t)/\bar{F}(t),\ \ t>0$, is the hazard rate function of X, $\bar{F}(t)=1-F(t)$, and $E_{f_X,\alpha}$ denotes the expectation with respect to the density function

$$
f_{X,\alpha}(x) = -\frac{\mathrm{d}\bar{F}^{\alpha}(x)}{\mathrm{d}x} = \alpha\bar{F}^{\alpha-1}(x)f(x), \qquad \alpha>0.
$$

It can be easily shown that $\lim_{\alpha\to 1} H_\alpha(X)=H(X)$, where

$$
H(X) = -\int_{-\infty}^{+\infty} f(x)\log f(x)\,\mathrm{d}x
$$

is commonly referred to as the entropy or *Shannon information measure* of X. The properties and virtues of $H(X)$ have been thoroughly investigated by Shannon (1948). A relatively recent reference for Shannon entropy is Cover and Thomas (1991).

Let θ be the parameter of one of the following families:

(i) location $F_\theta(x)=F_0(x-\theta)$, θ real;
(ii) scale $F_\theta(x)=F_0(\theta x)$, $\theta>0$.

Then in the location case the Rényi entropy is free of θ and for the case of scale family, it is a function of $-\log\theta$. This is also confirmed by Table 1 borrowed from Song (2001) which contains $H_\alpha(X)$ for some common distributions, e.g. exponential, Pareto, normal, Weibull and beta distributions. In that table $B(.,.)$ is the complete beta function. Nadarajah and Zografos (2003) also derived analytical formulas for Rényi entropy for 26 flexible families of univariate continuous distributions.

Table 1
Rényi entropy for some common distributions

Name	$f(x)$	$H_\alpha(X)$
Exponential	$\lambda e^{-\lambda x}$	$-\log\lambda - \frac{1}{1-\alpha}\log\alpha$
Pareto	$\lambda\beta^\lambda x^{-\lambda+1}$	$\log\beta + \frac{\alpha}{1-\alpha}\log\lambda - \frac{1}{1-\alpha}\log[(1+\lambda)\alpha - 1]$
Normal	$\frac{1}{\sqrt{2\pi}\sigma}\exp[-\frac{1}{2\sigma^2}(x-\mu)^2]$	$\log\sigma + \frac{1}{2}\log 2\pi - \frac{1}{2(1-\alpha)}\log\alpha$
Weibull	$\beta\lambda^\beta x^{\beta-1}\exp[-(\lambda x)^\beta]$	$-\log\lambda + \frac{1}{1-\alpha}\log\left[\frac{\Gamma\left(\alpha - \frac{\alpha-1}{\beta}\right)}{\alpha^\alpha}\right] - \log(\beta\alpha^\beta)$
Beta	$\frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$	$(1-\alpha)^{-1}\log\left[\frac{B(\alpha(a-1)+1,\alpha(b-1)+1)}{B^\alpha(a,b)}\right]$

Rényi entropies of order statistics $X_{1:n},\ldots,X_{n:n}$ are found by noting that $U_{i:n} = F_X(X_{i:n}),\ i=1,\ldots,n$, where $U_{i:n}$ is the ith order statistic from a random sample of size n from a Uniform (0, 1) distribution. The transformation formula for the Rényi entropy applied to $X_{i:n} = F_X^{-1}(U_{i:n})$ gives the following representations of the Rényi entropy of order statistics:

$$H_\alpha(X_{i:n}) = H_\alpha(U_{i:n}) + \frac{1}{1-\alpha}\log E[f^{\alpha-1}F^{-1}(W_i)], \tag{2}$$

where W_i has beta distribution with parameters $(i-1)\alpha+1$ and $(n-i)\alpha+1$.

Example 1. Suppose that X is a random variable having the exponential distribution with mean $1/\lambda$. Then $f(F^{-1}(t)) = \lambda(1-t)$ and thus we have

$$E[f^r(F^{-1}(W_i))] = \lambda^r\frac{B((i-1)\alpha+1,(n-i)\alpha+r+1)}{B((i-1)\alpha+1,(n-i)\alpha+1)}.$$

For the sample minimum, $i=1$, (2) gives

$$H_\alpha(X_{1:n}) = -\log\lambda - \log n - \frac{1}{1-\alpha}\log\alpha.$$

Thus, in this case, (2) confirms the fact that the sample minimum has an exponential distribution with parameter $n\lambda$. For the case of the sample maximum, $i=n$, we obtain

$$H_\alpha(X_{n:n}) = -\log\lambda + \frac{\alpha}{1-\alpha}\log n + \frac{1}{1-\alpha}\log B((n-1)\alpha+1,\alpha).$$

Numerical computations indicate that

$$\Delta(n,\alpha) = H_\alpha(X_{n:n}) - H_\alpha(X_{1:n}) = \frac{1}{1-\alpha}\log[n\alpha B((n-1)\alpha+1,\alpha)] \geqslant 0.$$

The fact that equality holds for $n=1$ is obvious. So the Rényi entropy of the maximum is always more than the Rényi entropy of the minimum in exponential samples. Also $\Delta(n,\alpha)$ is an increasing function of any of the arguments n and α, if the other argument is fixed.

3. Characterizations based on order statistics

In this section, we obtain some characterization results based on the Rényi entropy properties of order statistics. First, we recall two well-known identities.

- The relation between the hazard rate functions of $X_{1:n}$ and X is given by

$$r_{X_{1:n}}(t) = n r_X(t), \quad t > 0.$$

- The reversed hazard rate function of nonnegative random variable X is defined as

$$\tilde{r}_X(t) = \frac{f(t)}{F(t)}, \quad t > 0.$$

We know that $r_X(t)$ and $\tilde{r}_X(t)$ play important roles in the context of reliability theory. The following lemma is used in this paper. It is known in the literature as Müntz–Szász Theorem, which is often invoked in moment-based characterization theorems, see Kamps (1998), Borwein and Erdelyi (1995, Section 4.2).

Lemma 1. *For any strictly increasing sequence of positive integers* $\{n_j,\ j \geqslant 1\}$, *the sequence of polynomials* $\{x^{n_j}\}$ *is complete on* $L(0, 1)$ *if and only if* $\sum_{j=1}^{\infty} n_j^{-1} = \infty$.

In the sequel we assume that $\{n_j,\ j \geqslant 1\}$ is a strictly increasing sequence of positive integers.

Theorem 1. *Let X and Y be two random variables with pdfs* $f(x)$ *and* $g(x)$ *and absolutely continuous cdfs* $F(x)$ *and* $G(x)$, *respectively. Then F and G belong to the same family of distributions, but for a change in location and scale, if and only if*

$$H_\alpha(X_{m:n}) - H_\alpha(X) = H_\alpha(Y_{m:n}) - H_\alpha(Y),$$

for some fixed positive integer m and $n = n_j \geqslant m,\ j > 1$ *such that* $\sum_{j=1}^{+\infty} n_j^{-1}$ *is infinite.*

Proof. The necessity is trivial, hence it remains to prove the sufficiency part. By (1), we have

$$H_\alpha(X_{m:n}) - H_\alpha(X) = \frac{1}{1-\alpha}\left[\alpha A_{m,n} + \log \frac{\int_0^\infty [r(x)]^\alpha [F(x)]^{\alpha(m-1)} [\bar{F}(x)]^{\alpha(n-m+1)}\,\mathrm{d}x}{\int_0^\infty [r(x)]^\alpha [\bar{F}(x)]^\alpha\,\mathrm{d}x}\right],$$

where $A_{m,n} = \log n!/(m-1)!(n-m)!$. If for two cdfs F and G with corresponding pdfs f and g, respectively, these differences coincide, we can conclude that

$$\begin{aligned} &\frac{\int_0^1 [r_F(F^{-1}(1-w^{1/\alpha}))]^{\alpha-1}(1-w^{1/\alpha})^{\alpha(m-1)} w^{n-m}\,\mathrm{d}w}{\int_0^1 [r_F(F^{-1}(1-w^{1/\alpha}))]^{\alpha-1}\,\mathrm{d}w} \\ &\quad = \frac{\int_0^1 [r_G(G^{-1}(1-w^{1/\alpha}))]^{\alpha-1}(1-w^{1/\alpha})^{\alpha(m-1)} w^{n-m}\,\mathrm{d}w}{\int_0^1 [r_G(G^{-1}(1-w^{1/\alpha}))]^{\alpha-1}\,\mathrm{d}w}, \end{aligned} \tag{3}$$

where r_F is the hazard rate function of F. Let

$$c^{\alpha-1} = \frac{\int_0^1 [r_F(F^{-1}(1-w^{1/\alpha}))]^{\alpha-1}\,\mathrm{d}w}{\int_0^1 [r_G(G^{-1}(1-w^{1/\alpha}))]^{\alpha-1}\,\mathrm{d}w}.$$

Then, (3) can be expressed as

$$\int_0^1 \{[r_F(F^{-1}(1-w^{1/\alpha}))]^{\alpha-1} - [c r_G(G^{-1}(1-w^{1/\alpha}))]^{\alpha-1}\} \frac{(1-w^{1/\alpha})^{\alpha(m-1)}}{w^{m-n}}\,\mathrm{d}w = 0. \tag{4}$$

If (4) holds for $n = n_j \geqslant m,\ j \geqslant 1$, such that $\sum_{j=1}^{+\infty} n_j^{-1} = \infty$, then from Lemma 1 we can conclude that

$$r_F(F^{-1}(1-w^{1/\alpha})) = c r_G(G^{-1}(1-w^{1/\alpha})) \quad \text{a.e. } w \in (0,1)$$

or equivalently $f(F^{-1}(t)) = c g(G^{-1}(t))$ for all $0 < t < 1$. Since $\mathrm{d}(F^{-1}(t))/\mathrm{d}t = 1/f(F^{-1}(t))$, it then follows that $F^{-1}(t) = c G^{-1}(t) + d$. This means F and G belong to the same family of distributions, but for a change of location and scale. $\square$

In reliability theory, $X_{1:n}$, $X_{n:n}$ and $X_{n-m+1:n}$ represent the lifetimes of a series system, a parallel system and an m-out of-n system, respectively. In the following corollary, we can characterize these systems by putting 1, n and $n-m+1$ in place of m in Theorem 1.

Corollary 1. *Suppose the assumptions of Theorem* 1 *hold, then two systems A and B have the same lifetime distributions, but for a change in location and scale, if and only if one of the following statements holds*:

(i) *for series system* $H_\alpha(X_{1:n}) - H_\alpha(X) = H_\alpha(Y_{1:n}) - H_\alpha(Y)$,
(ii) *for parallel system* $H_\alpha(X_{n:n}) - H_\alpha(X) = H_\alpha(Y_{n:n}) - H_\alpha(Y)$, *and*
(iii) *for m-out of-n system* $H_\alpha(X_{n-m+1:n}) - H_\alpha(X) = H_\alpha(Y_{n-m+1:n}) - H_\alpha(Y)$, *for some fixed positive integer m* $n = n_j \geqslant m$, $j \geqslant 1$, *such that* $\sum_{j=1}^{+\infty} n_j^{-1}$ *is infinite.*

It is well-known that in the continuous case, only for the exponential distribution the hazard rate function is constant. Thus, by Corollary 1, we find the following results concerning the exponential distribution.

Corollary 2. *The family of exponential distributions with location and scale parameters* μ *and* σ, *respectively, i.e. with survival function* $\bar{F}(x) = \mathrm{e}^{-(x-\mu)/\sigma}$, *or with constant hazard rate, can be characterized by the condition*

$$H_\alpha(X_{1:n}) - H_\alpha(X) = -\log n,$$

for $n = n_j$, $j \geqslant 1$ *such that* $\sum_{j=1}^{+\infty} n_j^{-1}$ *is infinite.*

Gertsbakh and Kagan (1999) and Zheng (2001) obtained related characterizations of the Weibull family based on the properties of the Fisher information under types I and II censoring.

From part (ii) of Corollary 1, the independence of the reversed hazard rate function from x, can be characterized by the differences between the Rényi entropy of X and the Rényi entropy of the last order statistic. This is stated below.

Corollary 3. *The reversed hazard rate function* $\tilde{r}_X(t)$ *is constant, if and only if*

$$H_\alpha(X_{n:n}) - H_\alpha(X) = -\log n,$$

for $n = n_j$, $j \geqslant 1$ *such that* $\sum_{j=1}^{+\infty} n_j^{-1}$ *is infinite.*

It may be noted that the distribution $F(x) = \mathrm{e}^{\eta(\theta)x}$, $x < 0$ for some positive function $\eta(\theta)$, can be characterized under the assumptions of Corollary 3.

In the following theorem, we show that the parent distribution can be characterized by the Rényi entropy of $X_{m:n}$.

Theorem 2. *Under the assumptions of Theorem* 1, *F and G belong to the same location family of distributions, if and only if for a fixed m*,

$$H_\alpha(X_{m:n_j}) = H_\alpha(Y_{m:n_j}), \quad \forall n_j \geqslant m,$$

such that $\sum_{j=1}^{+\infty} n_j^{-1}$ *is infinite.*

Proof. The necessity is trivial, hence it remains to prove the sufficiency part. We have

$$H_\alpha(X_{m:n}) = \frac{1}{1-\alpha}\left[\alpha A_{m,n} + \log \int_0^\infty [r(x)]^\alpha [F(x)]^{\alpha(m-1)} [\bar{F}(x)]^{\alpha(n-m+1)}\, \mathrm{d}x\right],$$

where $A_{m,n} = \log n!/(m-1)!(n-m)!$. If for two cdfs F and G, $H_\alpha(X_{m:n}) = H_\alpha(Y_{m:n})$, we can conclude that

$$\int_0^1 \{[r_G(G^{-1}(1-w^{1/\alpha}))]^{\alpha-1} - [r_F(F^{-1}(1-w^{1/\alpha}))]^{\alpha-1}\}(1-w^{1/\alpha})^{\alpha(m-1)} w^{n-m}\, \mathrm{d}w = 0. \tag{5}$$

By taking $k = n-m$ in (5), this can be easily converted into a sequence of equations as given in Lemma 1. Thus we can conclude that $r_F(F^{-1}(t)) = r_G(G^{-1}(t))$ or equivalently $f(F^{-1}(t)) = g(G^{-1}(t))$ for all $0 < t < 1$. It then follows that $F^{-1}(t) = G^{-1}(t) + d$. This means F and G belong to the same family of distributions, but for a location shift. □

Using Theorem 2, we get the following corollary.

Corollary 4. *Under the assumptions of Theorem* 2, *two systems A and B have the same lifetime distributions*, *but for a change in location*, *if and only if one of the following statements holds*:

(i) *for series system* $H_\alpha(X_{1:n}) = H_\alpha(Y_{1:n})$,
(ii) *for parallel system* $H_\alpha(X_{n:n}) = H_\alpha(Y_{n:n})$, *and*
(iii) *for m-out of-n system* $H_\alpha(X_{n-m+1:n}) - H_\alpha(X) = H_\alpha(Y_{n-m+1:n}) - H_\alpha(Y)$, *for some fixed positive integer m and* $n = n_j \geqslant m$, $j \geqslant 1$, *such that* $\sum_{j=1}^{+\infty} n_j^{-1}$ *is infinite.*

Remark 1. By letting $\alpha \to 1$, the results of this section are seen to hold for Shannon entropy, as it was shown directly by Baratpour et al. (2007).

4. Characterizations based on entropy of record values

Let $U_1, U_2, \ldots$ be the sequence of upper record values produced by the sequence of X_i's from cdf F and pdf f, then the pdf of U_n is given by

$$f_{U_n}(u) = \frac{[-\log \bar{F}(u)]^{n-1}}{(n-1)!} f(u), \quad -\infty < u < +\infty, \tag{6}$$

see Arnold et al. (1998) for more details.

Basically, it is said that a component with lifetime X and distribution function $F(x)$ has been minimally repaired upon failure at time x_0 if the distribution function for its next failure is given by $(F(x) - F(x_0))/(1 - F(x_0))$, for all $x \geqslant x_0$. Let $X(n)$ denote the lifetime of the component if n minimal repairs are allowed, then the survival function of the $X(n)$ is the same as U_{n+1} (see, Shaked and Shanthikumar, 1994). Thus the study of the theory of record values is the same as the study of lifetimes with minimal repairs.

In this section, we show that F can be uniquely determined up to a location change by the equality of Rényi entropy of record values. First, we present the following lemma.

Lemma 2 (*Goffman and Pedrick, 1965, pp. 192–193*). *A complete orthonormal system for the space* $L_2(0, \infty)$ *is given by the sequence of Laguerre function*

$$\phi_n(x) = \frac{1}{n!} \mathrm{e}^{-x/2} L_n(x), \quad n \geqslant 0,$$

where $L_n(x)$ *is the Laguerre polynomial*, *defined as the sum of coefficients of* e^{-x} *in the nth derivative of* $x^n \mathrm{e}^{-x}$, *that is*

$$\begin{aligned} L_n(x) &= \mathrm{e}^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n \mathrm{e}^{-x}) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} n(n-1)\ldots(k+1)x^k. \end{aligned}$$

The meaning of the completeness of Laguerre functions in $L_2(0, \infty)$ *is that if* $f \in L_2(0, \infty)$ *and*

$$\int_0^{+\infty} f(x) \mathrm{e}^{-x/2} L_n(x)\, \mathrm{d}x = 0, \quad \forall n \geqslant 0,$$

then f is zero almost everywhere.

Theorem 3. *Suppose that the assumptions of Theorem* 1 *hold*, *moreover* $E(\log^2 f(X)) < +\infty$ *and* $E(\log^2 g(X)) < +\infty$. *Then F and G belong to the same location family of distributions*, *if and only if*

$$H_\alpha(U_n^X) = H_\alpha(U_n^Y), \quad \forall n \geqslant 1,$$

where U_n^X *and* U_n^Y *are the nth upper records of X and Y*, *respectively.*

Proof. The necessity is trivial, hence it remains to prove the sufficiency part. By (1) and (6) we have

$$H_\alpha(U_n^X) = \frac{1}{1-\alpha}\log\int_{-\infty}^{+\infty}\frac{(-\log\bar{F}(x))^{\alpha(n-1)}}{((n-1)!)^\alpha}f^\alpha(x)\,\mathrm{d}x.$$

If for two cdfs F and G, these differences coincide, we can conclude that

$$\int_0^\infty \mathrm{e}^{-w^{1/\alpha}}w^{1/\alpha-1}\{f^{\alpha-1}(F^{-1}(1-\mathrm{e}^{-w^{1/\alpha}}))-g^{\alpha-1}(G^{-1}(1-\mathrm{e}^{-w^{1/\alpha}}))\}w^{n-1}\,\mathrm{d}w=0, \tag{7}$$

for all $n \geqslant 1$. By (7), we can conclude that

$$\int_0^\infty w^{1/\alpha-1}\mathrm{e}^{w/2-w^{1/\alpha}}\{f^{\alpha-1}(F^{-1}(1-\mathrm{e}^{-w^{1/\alpha}}))-g^{\alpha-1}(G^{-1}(1-\mathrm{e}^{-w^{1/\alpha}}))\}\mathrm{e}^{-w/2}Ł_n(w)\,\mathrm{d}w=0,$$

for all $n \geqslant 1$, where $Ł_n(w)$ is Laguerre polynomial given in Lemma 2. Using the assumptions $E(\log^2 f(X)) < +\infty$ and $E(\log^2 g(X)) < +\infty$, and Minkowski inequality, we can conclude that

$$w^{1/\alpha-1}\mathrm{e}^{-(w/2-w^{1/\alpha})}\{f^{\alpha-1}(F^{-1}(1-\mathrm{e}^{-w^{1/\alpha}}))-g^{\alpha-1}(G^{-1}(1-\mathrm{e}^{-w^{1/\alpha}}))\}\in L^2(0,1).$$

Hence, by the completeness property alluded to Lemma 2, we can conclude that $f(F^{-1}(t)) = g(G^{-1}(t))$ for all $0 < t < 1$. Rest of the proof is similar to the proof of Theorem 2. Thus the result follows. □

Remark 2. Result similar to Theorem 3 holds for lower record values.

Remark 3. By letting $\alpha \to 1$, the results of this section are seen to hold for Shannon entropy, as it was shown directly by Baratpour et al. (2007).

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