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Wavelet based estimation of the derivatives of a density for a negatively associated process

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Abstract

We propose a method of estimation of the derivatives of probability density based wavelets methods for a sequence of negatively associated random variables with a common one-dimensional probability density function and obtain an upper bound on L_p-losses for the such estimators.

Key words and phrases: Negative dependence; Multiresolution analysis, Besov space, wavelets, nonparametric estimation of derivatives of a density

1 Introduction

Methods of estimation of density and regression function is quite common in statistical applications. Recently, there has been a lot of interest in nonparametric estimation of such functions based on wavelets. The reader may be referred to Härdle *et al.*(1998) and Vidakovic (1999) for a detailed coverage of wavelet theory in statistics and to Prakasa Rao (1999) for a recent comprehensive review and application of these and other methods of nonparametric functional estimation.

Antoniadis *et al.* (1994) and Masry (1994) among others discuss the estimation of regression and density function using the wavelets. Prakasa Rao (1996) considered the use of wavelets for estimating the derivatives of a density and investigated further their use for estimating the integrated squared density [see Prakasa Rao (1999a)]. Walter and Ghorai (1992) discuss the advantages and disadvantages of wavelet based methods of nonparametric estimation from *i.i.d.* sequences of random variables. Prakasa Rao (2003) echoes the same advantages and disadvantages for the case of associated sequences while dealing with nonparametric estimation of density itself using wavelets. It should be pointed out that these methods allow one to obtain precise limits on the asymptotic mean squared error for the estimator of density and its derivatives as well as some other functionals of the density [see Prakasa Rao (1996, 1999a)]. Recently, Chaubey *et al.* (2006) have generalized the results of Prakasa Rao (1996) to estimating the derivatives of a density for associated sequences. Similar results were also obtained by Doosti *et al.* (2005) in estimating the density itself for negatively associated sequences as a generalization of the results of Prakasa Rao (2003) for estimating the density for associated sequences. Here we generalize these results to the case of estimating derivatives of a density of negatively associated sequences. We recall the definition of negative association for an arbitrary collection of random variables.

Definition 1.1 *A finite family of random variables (r.v.s) $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA)if, for every pair of disjoint subsets A and B of $\{1, 2, ..., n\}$, we have*

$$Cov\{h_1(X_i, i \in A), h_2(X_j, j \in B)\} \leq 0,$$

whenever h_1 and h_2 are real-valued coordinate-wise increasing functions and the covariance exists. A random process $\{X_i\}_{i=-\infty}^{\infty}$ is NA if every finite sub-family is NA.

The dependence structure characterized by NA was first introduced by Alam and Saxena (1981) and later studied by Joag-Dev and Proschan (1983). It has found a number of applications in certain fields. Roussas (1996) provides an excellent review of the subject with a comprehensive list of references.

In this paper, our purpose is to extend the results in Prakasa Rao (1996) for estimating the derivatives of a density using wavelets to the case of a negatively associated sequence along the lines in Prakasa Rao (2003).

The organization of the paper is as follows. In section 2, we discuss the preliminaries of the wavelet based estimation of the derivatives of the density along with the necessary underlying setup considered in Prakasa Rao (1996). Then in section 3, we extend his result to squared integrated error measured in $p-$norm. This result is then generalized to NA case and finally we obtain bounds on the L_p-losses similar to the one obtained by Prakasa Rao (2003) for density estimation for the case of positive association.

2 Preliminaries

Let $\{X_n, n \geq 1\}$ be a sequence of random variables on the probability space $(\Omega, \aleph, P)$. We suppose that X_i has a bounded and compactly supported marginal density $f(.)$, with respect to Lebesgue measure, which does not depend on i. We estimate this density from n observations $X_i, i = 1, ..., n$. For any function $f \in \mathbf{L}_2(\mathbf{R})$, we can write a formal expansion (see Daubechies (1992)):

$$f = \sum_{k \in Z} \alpha_{j_0,k}\phi_{j_0,k} + \sum_{j \geq j_0} \sum_{k \in Z} \delta_{j,k}\psi_{j,k} = P_{j_0}f + \sum_{j \geq j_0} D_j f$$

where the functions

$$\phi_{j_0,k}(x) = 2^{j_0/2}\phi(2^{j_0}x - k)$$

and

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$$

constitute an (inhomogeneous) orthonormal basis of $\mathbf{L}_2(\mathbf{R})$. Here $\phi(x)$ and $\psi(x)$ are the scale function and the orthogonal wavelet, respectively. Wavelet coefficients are given by the integrals

$$\alpha_{j_0,k} = \int f(x)\phi_{j_0,k}(x)dx, \delta_{j,k} = \int f(x)\psi_{j,k}dx$$

We suppose that both ϕ and $\psi \in \mathbf{C}^r$, (space of functions with r continuous derivatives), r being a positive integer and have compact supports included in $[-\delta, \delta]$, for some $\delta > 0$. Note that, by corollary 5.5.2 in Daubechies (1988), ψ is orthogonal to polynomials of degree $\leq r$, *i.e.*

$$\int \psi(x)x^l dx = 0, \forall l = 0.1, ..., r$$

We suppose that f belongs to the Besov class (see Meyer (1990), §VI.10), $F_{s,p,q} = \{f \in B^s_{p,q}, \|f\|_{B^s_{p,q}} \leq M\}$ for some $0 \leq s \leq r+1, p \geq 1$ and $q \geq 1$, where

$$\|f\|_{B^s_{p,q}} = \|P_{j_0}f\|_p + (\sum_{j \geq j_0} (\|D_j f\|_p 2^{js})^q)^{1/q}$$

We may also say $f \in B^s_{p,q}$ if and only if

$$\|\alpha_{j_0,.}\|_{l_p} < \infty, \qquad \text{and} \qquad (\sum_{j \geq j_0} (\|\delta_{j,.}\|_{l_p} 2^{j(s+1/2-1/p)})^q)^{1/q} < \infty \tag{2.1}$$

where$\|\gamma_{j,.}\|_{l_p} = (\sum_{k\in Z} \gamma^p_{j,k})^{1/p}$. We consider Besov spaces essentially because of their executional expressive power [see Triebel (1992) and the discussion in Donoho *et al.* (1995)]. We construct the density estimator [see Prakasa Rao (2003)]

$$\hat{f} = \sum_{k \in K_{j_0}} \hat{\alpha}_{j_0,k} \phi_{j_0,k}, \qquad \text{with} \quad \hat{\alpha}_{j_0,k} = \frac{1}{n} \sum_{i=1}^{n} \phi_{j_0,k}(X_i), \tag{2.2}$$

where K_{j_0} is the set of k such that $supp(f) \cap supp(\phi_{j_0,k}) \neq \emptyset$. The fact that ϕ has a compact support implies that K_{j_0} is finite and $card(K_{j_0}) = O(2^{j_0})$. Wavelet density estimators aroused much interest in the recent literature, see Donoho *et al.* (1996) and Doukhan and Leon (1990). In the case of independent samples, the properties of the linear estimator (2.2) have been studied for a variety of error measures and density classes [see Kerkyacharian and Picard (1992), Leblanc (1996) and Tribouley (1995)]. In the setup considered by Prakasa Rao (1996), we assume that ϕ is a scaling function generating an $r-$regular multiresolution analysis and $f^{(d)} \in \mathbf{L}_2(\mathbf{R})$, for some $r \geq (d+1)$. Furthermore, we assume that there exists $C_m \geq 0$ and $\beta_m \geq 0$ such that

$$|f^{(m)}(x)| \leq C_m(1+|x|)^{-\beta_m}, 0 \leq m \leq r. \tag{2.3}$$

Prakasa Rao (1996) showed that the projection of $f^{(d)}$ on V_{j_0} is

$$f^{(d)}_{j_0}(x) = \sum_{k \in K_{j_0}} a_{j_0,k} \phi_{j_0,k}(x),$$

where

$$a_{j_0,k} = (-1)^d \int \phi^{(d)}_{j_0,k}(x) f(x) dx.$$

So its estimator is

$$\hat{f}^{(d)}_{j_0}(x) = \sum_{k \in K_{j_0}} \hat{a}_{j_0,k} \phi_{j_0,k}(x), \tag{2.4}$$

where

$$\hat{a}_{j_0,k} = \frac{(-1)^d}{n} \sum_{i=1}^{n} \phi^{(d)}_{j_0,k}(X_i).$$

The estimator in Eq. (2.4) will be used as an estimator for $f^{(d)}(x)$.

3 Main Results

First, we consider the sequence $\{X_i, i = 1, ..., n\}$ to be consisting of *i.i.d.* random variables and extend the result of Prakasa Rao (1996) to integrated squared error, when the error is measured in $p-$norm. Therefore, one obtains his result by letting $p = 2$. Also, by considering $d = 0$, we obtain the results obtained in Kerkyacharian and Picard (1992), Leblanc (1996) and Tribouley (1995). Next, we consider the case of sequences with NA and obtain similar results in Theorems 3.2 and 3.3; here an additional condition on the scale function, namely monotonicity, is imposed. In theorem 3.2 and 3.3, the results of Doosti *et al.*(2005) are obtained by letting $d = 0$.

Before we discuss the main theorems of this paper, we state the following results that will be required in subsequent proofs, which are readily obtained by using the results (1.6) and (1.7) of Theorem 2 of Shao (2000):

Let $\{\xi_i, 1 \leq i \leq n\}$ be a sequence of NA identically distributed random variables such that $\mathbf{E}(\xi_i) = 0$, and $\|\xi_i\|_\infty < M < \infty$. Then there exist positive constants $C_1(p)$ and $C_2(p)$ such that

$$\mathbf{E}(|\sum_{i=1}^{n} \xi_i|^p) \leq C_1(p)\{M^{p-2}\sum_{i=1}^{n}\mathbf{E}(\xi_i^2) + (\sum_{i=1}^{n}\mathbf{E}(\xi_i^2))^{p/2}\}, p > 2 \tag{3.1}$$

and

$$\mathbf{E}(|\sum_{i=1}^{n} \xi_i|^p) \leq C_2(p)\{M^{p-1}\sum_{i=1}^{n}\mathbf{E}|\xi_i|)\}, 1 < p \leq 2. \tag{3.2}$$

Theorem 3.1 *Let $f^{(d)}(x) \in F_{s,p,q}$ with $s \geq \max(1/p, d), p \geq 1,$ and $q \geq 1$. Consider the linear wavelet density estimator in Eq. (2.4) for an i.i.d. sequence of random variables $X_1, ..., X_n$. Then for $p' \geq max(2, p)$, there exists a constant C such that*

$$\mathbf{E}\|\hat{f}_{j_0}^{(d)}(x) - f^{(d)}(x)\|_{p'}^2 \leq C\ n^{-\frac{2(s'-d)}{1+2s'}}$$

where $s' = s + 1/p' - 1/p$ and $2^{j_0} = n^{\frac{1}{1+2s'}}$.

Proof: First, we decompose $\mathbf{E}\|\hat{f}_{j_0}^{(d)}(x) - f^{(d)}(x)\|_{p'}^2$ into a bias term and stochastic term

$$\mathbf{E}\|\hat{f}_{j_0}^{(d)}(x) - f^{(d)}(x)\|_{p'}^2 \leq 2(\|f^{(d)} - f_{j_0}^{(d)}\|_{p'}^2 + \mathbf{E}\|\hat{f}_{j_0}^{(d)} - f_{j_0}^{(d)}\|_{p'}^2) = 2(T_1 + T_2) \tag{3.3}$$

Now, we want to find upper bounds for T_1 and T_2.

$$\sqrt{T_1} = \| \sum_{j \geq j_0} D_j f^{(d)} \|_{p'} \leq \sum_{j \geq j_0} (\|D_j f^{(d)}\|_{p'} 2^{js'}) 2^{-js'}$$

$$\leq \{\sum_{j \geq j_0} (\|D_j f^{(d)}\|_{p'} 2^{js'})^q\}^{1/q} \{\sum_{j \geq j_0} 2^{-js'q'}\}^{1/q'}$$

By Holder's inequality, with $1/q + 1/q' = 1$, from the above equation, we have

$$T_1 \leq C\|f^{(d)}\|_{B^{s'}_{p',q}} 2^{-s'j_0} \leq C\|f^{(d)}\|_{B^s_{p,q}} 2^{-s'j_0}. \tag{3.4}$$

The last inequality holds,because of the continuous Sobolev injection [see Triebel (1992) and the discussion in Donoho *et al.*(1996)]which implies that for $B^s_{p,q} \subset B^{s'}_{p',q}$, one gets,

$$\|f^{(d)}\|_{B^{s'}_{p',q}} \leq \|f^{(d)}\|_{B^s_{p,q}}.$$

Therefore, we get from Eq. (3.4)

$$T_1 \leq K 2^{-2s'j_0} \tag{3.5}$$

Next, we have

$$T_2 = \mathbf{E}\|\hat{f}^{(d)}_{j_0} - f^{(d)}_{j_0}\|^2_{p'} = \mathbf{E}\| \sum_{k \in K_{j_0}} (\hat{a}_{j_0,k} - a_{j_0,k})\phi_{j_0,k}(x)\|^2_{p'}.$$

This gives by using Lemma 1 in Leblanc (1996), p. 82 (using Meyer (1990)),

$$T_2 \leq C\mathbf{E}\{\|\hat{a}_{j_0,k} - a_{j_0,k}\|^2_{l_{p'}}\} 2^{2j_0(1/2-1/p')}.$$

Further, by using Jensen's inequality the above equation implies,

$$T_2 \leq C 2^{2j_0(1/2-1/p')} \{\sum_{k \in K_{j_0}} \mathbf{E}|\hat{a}_{j_0,k} - a_{j_0,k}|^{p'}\}^{2/p'}. \tag{3.6}$$

To complete the proof, it is sufficient to estimate $\mathbf{E}|\hat{a}_{j_0,k} - a_{j_0,k}|^{p'}$. We know that

$$\hat{a}_{j_0,k} - a_{j_0,k} = \frac{1}{n} \sum_{i=1}^{n} \{[\phi^{(d)}_{j_0,k}(X_i) - a_{j_0,k}]\}.$$

Denote $\xi_i = [\phi^{(d)}_{j_0,k}(X_i) - a_{j_0,k}]$. Note that $\|\xi_i\|_\infty \leq K.2^{j_0(1/2+d)}\|\phi\|_\infty, \mathbf{E}\xi_i = 0, \mathbf{E}\xi_i^2 \leq \|f\|_\infty 2^{2j_0d}\int_{-\infty}^{\infty}\phi^{2(d)}(v)dv$ and $|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}| = \frac{1}{n}|\sum_{i=1}^n \xi_i|$. Hence applying the result in Eq. (3.1) and using $card(K_{j_0}) = O(2^{j_0})$ we have

$$\begin{aligned}\{\sum_{k\in K_{j_0}} \mathbf{E}|\hat{a}_{j_0,k} - a_{j_0,k}|^{p'}\}^{2/p'} &\leq \{C2^{j_0}\frac{1}{n^{p'}}(n2^{(j_0/2)(p'-2+2dp')}c_1 + n^{p'/2}2^{j_0dp'}c_2)\}^{2/p'}\\ &\leq K_1\{\frac{2^{j_0(1+2d)}}{n^{2(1-1/p')}} + \frac{2^{2j_0d+2(j_0/p\prime)}}{n}\}. \qquad (3.7)\end{aligned}$$

Now by substituting the above bound in (3.6), we get

$$\begin{aligned}T_2 &\leq K_1 2^{2j_0(1/2-1/p')}\{\frac{2^{j_0(1+2d)}}{n^{2(1-1/p')}} + \frac{2^{2j_0d+2(j_0/p')}}{n}\} = K_1\{\frac{2^{2j_0-2(j_0/p')+2j_0d}}{n^{2-2/p'}} + \frac{2^{j_0(1+2d)}}{n}\}\\ &= K_1\{\frac{2^{j_0}}{n}(\frac{2^{j_0(1+2d)}}{n})^{1-2/p'} + \frac{2^{j_0(1+2d)}}{n}\}.\end{aligned}$$

Since $n \geq 2^{j_0}$ and $1-2/p' \geq 0$ imply $(\frac{2^{j_0}}{n})^{1-2/p'} \leq 1$, we have the inequality

$$T_2 \leq \frac{K_2 2^{j_0(1+2d)}}{n}. \qquad (3.8)$$

By using the bounds obtained in (3.5) and (3.8), and choosing j_0 such that $2^{j_0} = n^{\frac{1}{1+2s'}}$ in (3.3), the theorem is proved. □

Now, in the rest of paper we consider $\{X_i\}$ as a NA sequence of random variables. We also consider the derivatives of scale function, say $\phi^{(d)}$, to be bounded variation (BV) function.

Theorem 3.2 *Let $\phi^{(d)}$ be BV, $f^{(d)}(x) \in F_{s,p,q}$ with $s \geq \max(1/p, d), p \geq 1$, and $q \geq 1$. Consider the linear wavelet based estimator in Eq. (2.4) for NA sequence of random variables $X_1, ..., X_n$. Then, for $p' > max(2, p)$, there exists a constant C such that*

$$\mathbf{E}\|\hat{f}^{(d)}_{j_0}(x) - f^{(d)}(x)\|^2_{p'} \leq C\ n^{-\frac{2(s'-d)}{1+2s'}}$$

where $s' = s + 1/p' - 1/p$ and $2^{j_0} = n^{\frac{1}{1+2s'}}$.

Proof: The proof is similar to the proof of Theorem 3.1 (see Doosti *et al.*(2006). We shall prove Eq. (3.7) will remain true. Since $\phi^{(d)}$ is BV, so it is the difference of two finite-valued monotone increasing function, say ϕ_1, ϕ_2, on $[-\delta, \delta]$ (De Barra (1974) Theorem 2. page 84), i.e., $\phi^{(d)} = \phi_1 - \phi_2$. Also we could define

$$\phi_{1(j_0,k)}(x) = 2^{j_0/2}\phi_1(2^{j_0}x - k), \quad \phi_{2(j_0,k)}(x) = 2^{j_0/2}\phi_2(2^{j_0}x - k)$$

so we have $\phi^{(d)}_{j_0,k} = \phi_{1(j_0,k)} - \phi_{2(j_0,k)}$. Furthermore If we define:

$$\begin{aligned} a_1 &= (-1)^d \int \phi_{1(j_0,k)}(x) f(x) dx, \\ a_2 &= (-1)^d \int \phi_{2(j_0,k)}(x) f(x) dx, \\ \xi_{1(i)} &= \phi_{1(j_0,k)} - a_1, \\ \xi_{2(i)} &= \phi_{2(j_0,k)} - a_2, \end{aligned}$$

then it is easy to see:

$$\begin{aligned} \mathbf{E}\xi_{1(i)} &= \mathbf{E}\xi_{2(i)} = 0 \\ \|\xi_{l(i)}\|_\infty &\leq K\|\xi_i\|_\infty \leq 2^{j_0(1/2+d)}\|\phi\|_\infty, l = 1, 2, \\ \mathbf{E}\xi^2_{l(i)} &\leq K\mathbf{E}\xi_i \leq \|f\|_\infty 2^{2j_0 d} \int_{-\infty}^{\infty} \phi^{2(d)}(v) dv, l = 1, 2, \end{aligned}$$

, where ξ_i defined before.
In view of the NA property of the sequence $\{X_n, n \geq 1\}$ and the monotonicity of the functions $\phi_{1(j_0,k)}$ and $\phi_{2(j_0,k)}$, it follows that the sequences $\{\xi_{1(i)}, i \geq 1\}$ and $\{\xi_{2(i)}, i \geq 1\}$ are also a sequences of NA random variables. Now by considering Eq. (3.1) and using following below inequality, we see the Eq. (3.7) remain true.

$$|\sum \xi_i|^{p'} \leq 2^{p'}(|\sum \xi_{1(i)}|^{p'} + |\sum \xi_{2(i)}|^{p'}). \tag{3.9}$$

The rest of proof is similar to the proof of Theorem 3.1. □

Now, suppose $1 < p' \leq 2$, the following theorem gives an upper bound for the expected loss $\mathbf{E}\|\hat{f}^{(d)}_{j_0} - f^{(d)}\|^{p'}_{p'}$.

Theorem 3.3 *Let $f^{(d)}(x) \in F_{s,p,q}$ with $s \geq \max(1/p, d), p \geq 1,$ and $q \geq 1$ then for $1 < p' \leq 2$. Consider the linear wavelet based estimator in Eq. (2.4) for NA sequence of random variables $X_1, ..., X_n$. There exists a constant C such that*

$$\mathbf{E}\|\hat{f}^{(d)}_{j_0}(x) - f^{(d)}(x)\|^{p'}_{p'} \leq C\ n^{-\frac{(2s'-1/2-d)(p'-1)-d}{1+2s'}}$$

where $s' = s + 1/p' - 1/p$ and $2^{j_0} = n^{\frac{1}{1+2s'}}$.

Proof: Observing that

$$\mathbf{E}\|\hat{f}_{j_0}^{(d)} - f^{(d)}\|_{p'}^{p'} \leq 2^{p'-1}(\|f^{(d)} - f_{j_0}^{(d)}\|_{p'}^{p'} + \mathbf{E}\|\hat{f}_{j_0}^{(d)} - f_{j_0}^{(d)}\|_{p'}^{p'}) \tag{3.10}$$

and

$$\|f^{(d)} - f_{j_0}^{(d)}\|_{p'}^{p'} \leq C_1 2^{-p's'j_0} \tag{3.11}$$

we have

$$\mathbf{E}\|\hat{f}_{j_0}^{(d)} - f_{j_0}^{(d)}\|_{p'}^{p'} \leq C2^{2j_0(p'/2-1)}\{\sum_{k\in K_{j_0}} \mathbf{E}|\hat{a}_{j_0,k} - a_{j_0,k}|^{p'}\}. \tag{3.12}$$

Thus, to complete the proof, it is sufficient to estimate $\mathbf{E}|\hat{a}_{j_0,k} - a_{j_0,k}|^{p'}$. Let $\xi_i = [\phi_{j_0,k}^{(d)}(X_i) - a_{j_0,k}]$. Because of NA property and monotonicity of scale functions $\phi_{1(j_0,k)}$ and $\phi_{2(j_0,k)}$ we know that $\{\xi_{1(i)}, i \geq 1\}$ and $\{\xi_{2(i)}, i \geq 1\}$ remains a sequence of NA random variables. Moreover

$$\begin{aligned} \mathbf{E}\xi_{1(i)} &= \mathbf{E}\xi_{2(i)} = 0 \\ \|\xi_{l(i)}\|_\infty &\leq K\|\xi_i\|_\infty \leq K.2^{j_0(1/2+d)}\|\phi\|_\infty, l = 1, 2, \\ \mathbf{E}\xi_{l(i)}^2 &\leq K\mathbf{E}\xi_i \leq K(\|f\|_\infty)^{1/2}2^{j_0 d}, l = 1, 2, \end{aligned}$$

We know $|\hat{\alpha}_{j_0,k} - \alpha_{j_0,k}| = \frac{1}{n}|\sum_{i=1}^n \xi_i|$. Hence by the results in (3.2), (3.9) and using $card(K_{j_0}) = O(2^{j_0})$ we have

$$\begin{aligned} \left\{\sum_{k\in K_{j_0}} \mathbf{E}|\hat{a}_{j_0,k} - a_{j_0,k}|^{p'}\right\} &\leq C2^{j_0}n^{-p'}2^{j_0(1/2+d)(p'-1)}2^{j_0 d}n \\ &= C2^{j_0[1+d+(1/2+d)(p'-1)]}n^{1-p'}. \end{aligned}$$

Now by substituting the above bound in (3.11), we get

$$\begin{aligned} \mathbf{E}\|\hat{f}_{j_0}^{(d)} - f_{j_0}^{(d)}\|_{p'}^{p'} &\leq C\ 2^{2j_0(p'/2-1)}2^{j_0[1+d+(1/2+d)(p'-1)]}n^{1-p'} \\ &= C\ 2^{j_0[p'-1+d+(1/2+d)(p'-1)]}n^{1-p'} \\ &= C\ 2^{j_0[(3/2+d)(p'-1)+d]}n^{1-p'} \\ &= n^{\frac{[(3/2+d)(p'-1)+d]}{1+2s'}+1-p'} \\ &= n^{-\frac{(2s'-1/2-d)(p'-1)-d}{1+2s'}} \end{aligned}$$

Thus, we obtain the desired result. □

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