37-th Anuall Iranian Mathematics Conference
11-14 Shahrivar 1385, 2-5 September 2006 , pp. 195–198

Dense joint continuity of separately continuous mappings

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Abstract: Let X be a topological space. In this note, we are interested to find conditions on X such that for every compact space Y and separately continuous mapping $f : X \times Y \to Z$ is jointly continuous on $D \times Y$, where dense D is a dense subset of X.

1 Introduction

Let $f : X \times Y \to Z$ be a separately continuous mapping. I. Namioka [3] proved that if X is strongly countably complete and Y is a compact space. It was expected that the result of Namioka remain valid for arbitrary Baire space X and compact space Y. However, Talagrand [5] provided an example of a separately continuous mapping $f : X \times Y \to R$, where X is an α-favorable (hence it is Baire) and Y is compact such that for each point $x \in X$, f is not jointly continuous in $\{x\} \times Y$. The result of Talagrand raises the following question:
What are Baire spaces X such that for every compact space Y and separately continuous mapping $f : X \times Y \to R$ must be jointly continuous at each point of some set $D \times Y$, where D is a dense G_δ subset of X?

2 Results

Let X and Y be topological spaces, we say that the relation $N(X, Y)$ is satisfied if for every separately continuous map $F : X \times Y \to R$ there exists a dense G_δ subset D of X, such that F is jointly continuous on $D \times Y$.
A topological space X satisfies the property $\mathcal{N}$, if $N(X, K)$ is satisfied for every compact space Y.
A topological space K satisfies the property $\mathcal{N}^*$ if $N(X, K)$ is satisfied for

every Baire space X.
Let (X,τ) be a topological space and $\mathcal{P}$ a family of subsets of X. The game $\mathcal{G}_{\mathcal{P}}$ between two players α and β is done as follows:
Player β starts the game by selecting a nonempty open set V_1 of X; then player α chooses a non-empty open set $U_1 \subset V_1$ and a $P_1 \in \mathcal{P}$. When (V_i, U_i, P_i), $1 \leq i \leq n-1$, have been defined, player β picks a nonempty open set $V_n \subset U_{n-1}$ and α answers to his/her move by selecting a nonempty open set $U_n \subset V_n$ and $P_n \in \mathcal{P}$. In this way two players generate a sequence of nonempty open subsets of X

$$V_1 \supset U_1 \supset V_2 \cdots \supset V_n \supset U_n \ldots$$

and a sequence $\{P_n\}$ of elements of $\mathcal{P}$.
The player α wins the game $G_{\mathcal{P}}$ if $(\bigcap_{n=1}^{\infty} V_n) \bigcap (\overline{\bigcup_{n=1}^{\infty} P_n}) \neq \emptyset$. Otherwise the player β is said to have won the play.
By a strategy for player α in $G_{\mathcal{P}}$, we mean a sequence of mappings $s = \{s_n\}$ which is defined inductively as follows:
The domain of s_1 is the set of all open sets and it assigns to each nonempty open subset U of X a pair (V, P), where V is a nonempty open subset U and $P \in \mathcal{P}$. The domain of s_n is the set of all partial plays $(V_1, s_1(V_1), \ldots, V_n)$ and it assigns to such a partial play a pair (W_n, P_n), where W_n is an open subset of V_n and $P_n \in \mathcal{P}$. We say that the player α has a winning strategy for the game G_P if there exists a strategy s, such that α wins all plays provided that he/she acts according to strategy s. In this case, we say that X is an α-favorable space for the game $G_{\mathcal{P}}$, otherwise X is said to be an α-unfavorable space for this game. Similarly, winning strategy for the player β, β-favorablity are defined.
i) The game G of Choquet (Banach-Mazur)(see [1] and [2]) is the game $G_{\mathcal{P}}$ for $\mathcal{P} = \{X\}$.
ii)The game $G_{\mathcal{P}}$ of Christensen- Saint Raymond (see [1]and [4]) is the game $G_{\mathcal{P}}$ for $\mathcal{P} = \{\{x\} : x \in X\}$, the set of all singletons of X.
iii)The game G_a of Debs [2]is the game $G_{\mathcal{P}}$ with $\mathcal{P}$ is the family of K-analytic subsets of X.
Let $\{\mathcal{A}_n\}$ be a sequence of open families in the space X. The sequence $\{\mathcal{A}_n\}$ is said to be strongly countably complete if a decreasing sequence $\{F_n\}$ of closed subsets of X has nonempty intersection provided that each F_n is contained in some $A_m \in \mathcal{A}_m$. X is said to be strongly countably complete if there exists a strongly countably complete sequence of open coverings of X.
Namioka [3] proved the following:

THEOREM 2.1 *Let X be a strongly complete regular space, then X has the $\mathcal{N}$ property.*

Using topological games, Christensen [1] extended Namioka's theorem:

THEOREM 2.2 *If X is an α-favorable space for G_P, then X has the $\mathcal{N}$ prop-*

erty.

To see that the above result is in fact a generalization of Namioka's theorem, we need the following elementary result:

THEOREM 2.3 *Every strongly complete space is α-favorable.*

Proof. Let X be a strongly complete space and $\{\mathcal{A}_n\}$ be its associated family of open coverings of X. For each $A \subset X$, define

$$n(A) = \sup\{k : \forall n \leq k, \exists U_n \in \mathcal{A}_n : A \subset U_n\}.$$

Let β start a game by choosing an open nonempty subset U_1 of X. Let V_1 be an open subset of U_1 such that $\overline{V}_1 \subset U_1$ and $n(\overline{V}_1) \geq n(U_1) + 1$, choose any point $x_1 \in V_1$, then (V_1, x_1) would be the answer of player α to β.
In general, if $U_1, \ldots, U_n$ and $(V_1, x_1), \ldots, (V_{n-1}, x_{n-1})$ have already be selected by the two players. Let V_n be an open subset of U_n such that $\overline{V}_n \subset U$ and $n(\overline{V}_n) \geq n(U_n) + 1$ and choose a point $x_n \in V_n$. Then (V_n, x_n) would be the next choice of α-player. In this way, a strategy for α-player would be determined. Let $F_n = \overline{\{x_m : m \geq n\}}$, then $\{F_n\}$ has the finite intersection property. Since $n(\overline{V}_n) \geq n$, it follows that there is some $A_n \in \mathcal{A}_n$ such that $V_n \subset A_n$ and therefore $F_n \subset A_n$. Thus $\bigcap_{n\geq 1} F_n \neq \emptyset$ and therefore $\{x_n\}$ has a limit point in $\bigcap_{n\geq 1} V_n$. Hence α has a winning strategy and X is α-favorable.
Later, Saint-Reymond and Debs extended Christensen's result by proving that:

THEOREM 2.4 (**Saint-Reymond** [4]).*Every β-unfavorable space for the game G_P has the property $\mathcal{N}$.*

THEOREM 2.5 (**Debs** [2]).*Every β-unfavorable space for the game G_a has the $\mathcal{N}$ property.*

Here, we give outline of the proof of Theorem 2.5. To begin with, we need the following basic lemma:

LEMMA 2.6 *Let X be a Baire space, K a compact space, and $B \subset C_p(K)$. We define $E : B \times K \to \mathbf{R}$ by $E(\varphi, y) = \varphi(y)$, the evaluation map, and assume the set $\{E^y : y \in K\}$ is a Corson compact subset of $C_p(B)$. Furthermore let $\Phi : X \times K \to [0,1]$ be separately continuous function and assume that $\varepsilon > 0$ is such that the set*

$$S = \{x \in X : \exists \varphi \in B \text{ with } ||\Phi_x - \varphi|| \leq \frac{\varepsilon}{4}\}$$

contains a dense G_δ-set in X. Then the set

$$G_\varepsilon = \bigcup\{U : U \text{ is open in } X \text{ and } ||\Phi_{X_1} - \Phi_{X_2}|| \leq \varepsilon \text{ for all } x_1, x_2 \in U\}$$

is dense in X.

Outline of the proof of theorem 2.5 Let X be β-unfavorable for the game G_a, K a compact space and $\Phi : X \times K \to [0,1]$ a separately continuous function. We define $\hat{\Phi} : X \to C_p(K)$ by $\hat{\Phi}(x) = \Phi_x$. Let $\varepsilon > 0$ and assume that $\dim \hat{\Phi}(U) \geq \varepsilon$ for every nonempty open set U in X.
For $k \in \mathbf{N}$, we define

$$\Gamma_k : C([0,1]^k) \times (C_p(K,[0,1]))^k \to C_p(K)$$

by $\Gamma_k(f, \varphi_1, \ldots, \varphi_k) = f(\varphi_1, \ldots, \varphi_k)$. These functions are continuous. For an analytic set A, we set

$$\Gamma(\hat{\Phi}(A)) = \bigcup_{k=1}^{\infty} \Gamma_k(C([0,1]^k) \times (\hat{\Phi}(A)^k),$$

and we note that $\Gamma(\hat{\Phi}(A))$ is a K-analytic subset of K. For an open nonempty set U and A a K-analytic subset of X, the set

$$V(U,A) = \{x \in U : ||\Phi_x - \varphi|| > \frac{\varepsilon}{3} \text{ for } \varphi \in \Gamma(\hat{\Phi}(A))\}$$

is not of the first category. The proof of this assertion uses Lemma 2.6. The proof will be complete by devising a strategy for player β for the game G_a. The basic used here is that a K-Souslin set in X has the Baire property.

References

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