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φ-FACTORABLE OPERATORS

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ABSTRACT. This paper is an investigation of factorable operators on G with respect to a function-valued inner product, the so called φ-bracket product, on $L^2(G)$ where G is a locally compact abelian group and φ is a topological isomorphism on G.

1. INTRODUCTION AND PRELIMINARIES

In [12] we have defined the φ bracket product as a function-valued inner product on $L^2(G)$, where G is a locally compact abelian (LCA) group and φ is a topological isomorphism on G. The φ-bracket product as a new inner product on $L^2(G)$ is applicable to extend many ideas and constructions from the theory of factorable operators and Weyl-Heisenberg frames on $\mathbb{R}^n$, to the setting of LCA groups in a more general and different way.

We define and investigate φ-factorable operators.

Here we give some of the basics regarding LCA groups. For a comprehensive account of LCA groups we refer to [7, 10]. Suppose G is a LCA group with the Haar measure dx. A subgroup L of G is called a uniform lattice if it is discrete and co-compact (i.e G/L is compact). Let φ be a topological isomorphism on G. If L is a uniform lattice in G, then so is $\varphi(L)$. Indeed, obviously $\varphi(L)$ is discrete. Also by [10, Theorem 5.34] $G/\varphi(L)$ is topologically isomorphic to G/L and so it is compact. In this paper we always assume that $G/\varphi(L)$ is normalized i.e. $|G/\varphi(L)| = 1$. Denote by $\varphi(L)^\perp$ the annihilator of $\varphi(L)$ in $\hat{G}$, i.e. $\varphi(L)^\perp = \{\gamma \in \hat{G};\ \gamma(\varphi(L)) = \{1\}\}$, which is a uniform lattice in $\hat{G}$ (see [13, 14, 15]).

Let L be a uniform lattice in G. Choosing the counting measure on L, a relation between the Haar measures dx on G and $d\dot{x}$ on $G/\varphi(L)$ is given by the following

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special case of Weil's formula [7]:
For $f \in L^1(G)$, we have $\sum_{k\in L} f(x\varphi(k^{-1})) \in L^1(G/\varphi(L))$ and

$$\int_G f(x)dx = \int_{G/\varphi(L)} \sum_{\varphi(k^{-1})\in\varphi(L)} f(x\varphi(k^{-1}))d\dot{x}, \tag{1.1}$$

where $\dot{x} = x\varphi(L)$.

Let $f, g \in L^2(G)$. The φ-bracket product of f, g is defined by

$$[f, g]_\varphi(\dot{x}) = \sum_{k\in L} f\overline{g}(x\varphi(k^{-1})), \tag{1.2}$$

for all $x \in G$. We define the φ-norm of f as $\|f\|_\varphi(\dot{x}) = ([f, f]_\varphi(\dot{x}))^{1/2}$. In the sequel we collect several basic properties of the φ-bracket product which follow by easy direct computations. The reader who does not want to check the details is referred to [12]. Let $f, g \in L^2(G)$. Then $|[f, g]_\varphi| \leq \|f\|_\varphi \|g\|_\varphi$ (Cauchy Schwarz Inequality). Also obviously (1.1) implies $\int_{G/\varphi(L)} [f, g]_\varphi(\dot{x})d\dot{x} = \langle f, g \rangle_{L^2(G)}$. For $\gamma \in \hat{G}$, denote by M_γ the modulation operator on $L^2(G)$, i.e. $M_\gamma f(x) = \gamma(x)f(x)$, for all $f \in L^2(G)$. Then for $f, g \in L^2(G)$ and $\gamma \in \varphi(L)^\perp$ we have the following relation between the φ-bracket product and the usual inner product in $L^2(G)$:

$$\widehat{[f, g]_\varphi}(\gamma) = \langle f, M_\gamma g \rangle_{L^2(G)}. \tag{1.3}$$

We say $g \in L^2(G)$ is φ-bounded if there exists $M > 0$ so that $\|g\|_\varphi \leq M$ a.e.. For $f, g \in L^2(G)$ the function $[f, g]_\varphi g$ need not generally be in $L^2(G)$. But we have

Proposition 1.1. *If $f, g, h \in L^2(G)$ and g, h are φ-bounded then $[f, g]_\varphi h \in L^2(G)$.*

A sequence $(g_n)_{n\in\mathbb{N}} \subset L^2(G)$ is called φ-orthonormal if $[g_n, g_m]_\varphi = 0$, for all $n \neq m \in \mathbb{N}$ and $\|g_n\|_\varphi = 1$ for all $n \in \mathbb{N}$. Let $f \in L^2(G)$ and $(g_n)_{n\in\mathbb{N}}$ be a φ-orthonormal sequence in $L^2(G)$. An extension of [4, Theorem 4.13] from $\mathbb{R}$ to the setting of a LCA group gives Bessel's Inequality for φ-bracket products as follows:

$$\sum_{n\in\mathbb{N}} |[f, g_n]_\varphi(\dot{x})|^2 \leq \|f\|_\varphi^2(\dot{x}), \quad \text{for a.e. } \dot{x} \in G/\varphi(L). \tag{1.4}$$

A φ-orthonormal sequence $(g_n)_{n\in\mathbb{N}}$ is called a φ-orthonormal basis if $[f, g_n]_\varphi = 0$ a.e. for all $n \in \mathbb{N}$, implies $f = 0$ a.e. Let $(g_n)_{n\in\mathbb{N}}$ be a φ-orthonormal sequence. It is not difficult to mimic the standard proofs for a usual orthonormal sequence in a Hilbert space to obtain equivalent conditions for $(g_n)_{n\in\mathbb{N}} \subseteq L^2(G)$ to be a φ-orthonormal basis.

Proposition 1.2. *If $(g_n)_{n\in\mathbb{N}}$ is a φ-orthonormal sequence in $L^2(G)$, the following are equivalent.*

1. *$(g_n)_{n\in\mathbb{N}}$ is a maximal φ-orthonormal sequence, i.e. $(g_n)_{n\in\mathbb{N}}$ is not contained in any other φ-orthonormal set.*
2. *$(g_n)_{n\in\mathbb{N}}$ is a φ-orthonormal basis.*
3. *For each $f \in L^2(G)$, $f = \sum_{n\in\mathbb{N}} [f, g_n]_\varphi g_n$ a.e.*
4. *$\|f\|_\varphi^2 = \sum_{n\in\mathbb{N}} |[f, g_n]_\varphi|^2$ a.e. for all $f \in L^2(G)$ (Parseval Identity).*

(5) $\{M_\gamma g_n\}_{n\in\mathbb{N},\gamma\in\varphi(L)^\perp}$ *is an orthonormal basis for* $L^2(G)$.

Thanks to Zorn's Lemma and Proposition 1.2, $L^2(G)$ admits a φ-orthonormal basis.

2. φ-Factorable Operators

Throughout this paper we always assume that G is a second countable LCA group, φ is a topological isomorphism on G and the notation are as in Section 2.

A function $h \in L^\infty(G)$ is said to be φ-periodic if $h(x\varphi(k)) = h(x)$ for every $k \in L$, $x \in G$.

Definition 2.1. We say an operator $U : L^2(G) \to L^p(E)$, $1 \le p \le \infty$, is φ-factorable if $U(hf) = hU(f)$ for all $f \in L^2(G)$ and all φ-periodic $h \in L^\infty(G)$, where E is a subgroup of G or $G/\varphi(L)$.

A bounded operator U is φ-factorable if and only if it commutes with modulations. More precisely:

Lemma 2.2. *Let U be a bounded operator from $L^2(G)$ to $L^2(E)$, where E is a subgroup of G or $G/\varphi(L)$. U is φ-factorable if and only if*

$$U(M_\gamma g) = M_\gamma U(g) \text{ for all } g \in L^2(G),\ \gamma \in \varphi(L)^\perp. \tag{2.1}$$

Our main goal in this section is to characterize φ-factorable operators $U : L^2(G) \to L^p(G/\varphi(L))$, for $p = 1$ and $p = 2$.

Clearly the operator U defined by $U(f) = [f, g]_\varphi$ for $f \in L^2(G)$, is φ-factorable. We will also show that every φ-factorable operator $U : L^2(G) \to L^1(G/\varphi(L))$ is of this form. First we establish a lemma in which we show that two φ-factorable operators are equal on $L^2(G)$ if and only if their integrals over $G/\varphi(L)$ are the same.

Lemma 2.3. *Let $U_1, U_2 : L^2(G) \to L^1(G/\varphi(L))$ be two φ-factorable operators. Then $U_1 = U_2$ if and only if $\int_{G/\varphi(L)} U_1(f)(\dot{x})d\dot{x} = \int_{G/\varphi(L)} U_2(f)(\dot{x})d\dot{x}$, for every $f \in L^2(G)$.*

Now we have the following Riesz Representation Theorem which characterizes all φ-factorable operators from $L^2(G)$ to $L^1(G/\varphi(L))$.

Theorem 2.4. *A bounded operator $U : L^2(G) \to L^1(G/\varphi(L))$ is φ-factorable if and only if there exists $g \in L^2(G)$ such that $U(f) = [f, g]_\varphi$ a.e. for all $f \in L^2(G)$. Moreover $\|U\| = \|g\|$.*

The following theorem characterizes φ-factorable operators from $L^2(G)$ to $L^2(G/\varphi(L))$.

Theorem 2.5. *A bounded operator $U : L^2(G) \to L^2(G/\varphi(L))$ is φ-factorable if and only if there exists a φ-bounded $g \in L^2(G)$ such that $U(f) = [f, g]_\varphi$ a.e. for all $f \in L^2(G)$. Moreover $\|U\|^2 = ess\ sup_{\dot{x}\in G/\varphi(L)} \|g\|_\varphi^2(\dot{x})$.*

Next we show that every bounded φ-factorable operator on $L^2(G)$ is adjointable.

Proposition 2.6. *Let $U : L^2(G) \to L^2(G)$ be a bounded φ-factorable operator and U^* be its adjoint. Then U^* is φ-factorable. Moreover,*

$$[U(f), g]_\varphi = [f, U^*(g)]_\varphi, \quad a.e. \text{ for all } f, g \in L^2(G). \tag{2.2}$$

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