

Notes Due to Lorenz Curve and Lorenz Ordering in View of Weighted Distributions

G. R. Mohtashami Borzadaran¹ ,and Zahra Behdani²

¹*Department of Statistics, Ferdowsi University of Mashhad*

²*Department of Statistics, University of Birjand*

Abstract: The Lorenz curve is an important tool for analysis the size distribution of income and wealth. An income distribution F is preferred to an income distribution G in the sense of Lorenz order (generalized Lorenz order), if its Lorenz curve (generalized Lorenz curve), is nowhere below the Lorenz curve (generalized Lorenz curve), of G . Let \mathbb{L} denote the set of all non-negative random variables with finite and positive means, for $X \in \mathbb{L}$ with distribution F_X and quantile function $F_X^{-1}(u) = \{x : F_X(x) \leq u\}, u \in [0, 1]$, the Lorenz curve is given by $L_X(p) = \frac{1}{E(X)} \int_0^p F_X^{-1}(u) du, p \in [0, 1]$. The most widely used alternative to the Lorenz order is the generalized Lorenz order that we need define, generalized Lorenz curve ($GL_X(\cdot)$), where $GL_X(p) = E(X)L_X(p)$. The concepts such as Lorenz partial order, generalized Lorenz partial order, dilation order and second-order absolute Lorenz order are reviewed and discussed as seen in (Ramos and Sordo 2003). In Bartoszewicz and Skolimowska (2006) used representation in view of weighted distributions by the Lorenz curve via the idea of Jain et al (1989) that the length biased distribution is closely related to the Lorenz curve. In view of the weighted distributions :

(i) It is shown that how to derive and determine Characterization results related to Lorenz ordering, generalized Lorenz ordering, dilation order, and second-order absolute Lorenz order for the cases that weights are increasing or decreasing functions is obtained. For special cases such as, probability weighted moments, order statistics, proportional hazard rate, reversed proportional hazard rate, generalized version of records, upper and lower records, truncated distributions, renewal equilibrium distributions, hazard rate and reversed hazard rate are some simple weights that the above ordering aspects obtained in this paper.

(ii Increasing proportional likelihood ratio (IPLR) or decreasing proportional likelihood ratio (DPLR) properties are sufficient conditions for the Lorenz ordering

Corresponding author, email: gmb1334@yahoo.com

of truncated distributions and their connection with a few cases to weighted distribution is obtained. Preservation of IFRA (DFRA) and NBW (NWU) classes under weighting is discussed in Bartoszewicz and Skolimowska (2006), some of the results applied for special cases of weights. Also, their results applied in view of the generalized Lorenz curves.

Keywords: Weighted Distribution, Lorenz Curve, Lorenz Ordering, Generalized Lorenz Ordering, IFRA, DFRA, NBU, NWU, Probability Weighted Moments, Dispersion Ordering.

1 Introduction

100 years ago, in June 1905, a short article titled "Methods of measuring the concentration of wealth" published in JASA, proposing a simple method. It is called the Lorenz curve for visualizing distribution of income or wealth with respect to inequality or concentration. After this, huge of publications is obtained and reveal that the Lorenz curve is an important tool for analysis the size of the income and wealth. Among the applications of the Lorenz curve we will concentrate on Lorenz ordering and generalized Lorenz order and its connection with economics. Arnold (1987) proposed a class of Lorenz curve ordered with respect to indexing parameter and Sarabia et al (1999) proposed a family of Lorenz curve that can be ordered in a large number of cases. Kleiber (2003) surveyed selected applications of the Lorenz curve and related stochastic order in economics and econometrics with a bias towards problems in statistical distribution theory. These include characterizations of income distribution in terms of families of inequality measures, Lorenz ordering of multi parameter distributions in terms of their parameters, probability inequalities for distributions of quadratic forms. Based on the Lorenz ordering, if two distribution functions have associated Lorenz curves which do not intersect, then they can be ordered without ambiguity in terms of welfare functions which are symmetric, increasing and quasi-concave. Shorrocks (1983) and Kakwani (1984) introduced gen-

eralized Lorenz curves and generalized Lorenz ordering for the case that the Lorenz curves are intersected. Thistle (1989) showed that a distribution is uniquely determined by its generalized Lorenz curve. It is well-known from Atkinson (1970), Shorrocks (1983) and Kakwani (1984) that the generalized Lorenz ordering allow important judgments concerning economics welfare.

In this paper, we study the relationship between the weighted distributions and the parent distributions in the context of Lorenz curve, generalized Lorenz curve, Lorenz ordering, generalized Lorenz ordering. These relationship depends on the nature of the weighted function and give rise to interesting connections.

2 Weighted Distributions

On considering weighted function $w(x, \beta)$ which is a non-negative function with parameter β represent a family of distributions with pdf, $g_w(x, \beta, \theta) = \frac{w(x, \beta)f(x, \theta)}{E[w(X, \beta)]}$, which is called a weighted version of distribution. Let X be a random variable with pdf $f(x)$ and $w(x)$ be a non-negative function with $E(w(X)) \neq 0 < \infty$, Rao (1965) defined the weighted distribution of X with the weight function $w(\cdot)$ as a distribution having $g_w(x) = \frac{w(x)f(x)}{E(w(X))}$. The weighted distribution with $w(x) = x^k$, k positive integer, is called the length-sized biased of order k distribution. If $E(X) < \infty$, we have for distribution $\hat{f}(x) = \frac{xf(x)}{E(X)}$, $x > 0$, as the length biased or sized biased distribution associated with F such that $\hat{F}(x) = \frac{\int_0^x tf(t)dt}{E(X)}$. The results due to many aspects of statistical inference can be generalized based on various weights. Known aging properties of life distributions can be obtained via weighting distributions that can be seen in Jian et al (1989). Order statistics, record value, residual lifetime of a stationary renewal process, selection samples, hazard rate, reversed hazard rate, proportional hazard model, reversed proportional hazard model, Lorenz curve and probability weighted moments are some special cases of weighted families. Bartoszewicz and Skolimowska (2006) studied re-

lation of weighted distribution with classes of life distribution and used a representation of weighted distributions by Lorenz curve to obtain some results concerning their relation with life distributions.

- The weight $w(x) = e^{lx}x^i[F(x)]^j[\bar{F}(x)]^k$ is one that implies many famous weights. If $(i = j = k = 0)$, $(l = j = k = 0)$, $(i = l = 0, k = n - j)$, $(i = l = k = 0)$ and $(i = j = l = 0)$, then $w(\cdot)$ is moment generating function, moments, order statistics, reversed proportional hazard and proportional hazard respectively.
- The weight $w(x) = [-\ln F(x)]^j[-\ln \bar{F}(x)]^k$ is another weight that $(j = 0)$ and $(k = 0)$ implies upper record and lower record respectively.
- Let $A(x) = E[w(X)/X > x]$, then, $\bar{G}(x) = \frac{1}{E(w(X))}\bar{F}(x)A(x)$.

3 Lorenz Curve for Weighted Distributions

Let \mathbf{L} denote the set of all non-negative random variables with finite and positive means. For $X \in \mathbf{L}$ with distribution F_X and quantile function $F_X^{-1}(u) = \{x : F_X(x) \leq u\}$, $u \in [0, 1]$, the Lorenz curve is given by

$$L_X(p) = \frac{1}{E(X)} \int_0^p F_X^{-1}(u) du, p \in [0, 1].$$

It is increasing, convex, continuous on $p \in [0, 1]$ with $L(0) = 0$ and $L(1) = 1$. Also, $L_X(p) \leq p$, $\bar{L}_X(p) = 1 - L_X(p)$ and any function possessing these properties is the Lorenz curve of a certain statistical distribution (Thompson 1976). It is also worth noting that the Lorenz curve itself may be considered as an cdf on unit interval. The Zenga curve is given by $Z_X(p) = 1 - \frac{F_1^{-1}(p)}{F_1^{-1}(1)}, p \in [0, 1]$, where $F_1^{-1}(p) = \inf\{x : F_1(x) \geq p\}$ and $F_1(x) = \frac{1}{E(X)} \int_0^x tf(t)dt$. The concentration curve takes values from point $(0, 1)$ to $(1, 0)$ but does not have the behaviour of the Lorenz curve. The Lorenz curve and Zenga curve provided partial ordering. Here, we are interested to Lorenz ordering but in the cases that the Lorenz curves intersected, we need to use

generalized lorenz order. It is defined in terms of generalized Lorenz curve , $GL_X(u)$ where

$$GL_X(p) = E(X)L_X(p) = \int_0^p F_X^{-1}(u)du, p \in [0, 1].$$

Generalized Lorenz curves are non-decreasing, continuous and convex with $GL_X(0) = 0$ and $GL_X(1) = E(X)$. A distribution is uniquely determined by its generalized Lorenz curve.

- $A_X(p) = \int_0^p [F^{-1}(t) - E(X)]dt, 0 \leq p \leq 1$ is called the absolute Lorenz curve, and is used in economics to compare income distributions. It is decreasing for $0 \leq p \leq F(E(X))$ and increasing for $F(E(X)) < p \leq 1$ and it takes values $A_X(0) = A_X(1) = 0$, and is convex function with respect to p .

It is interesting to note that the length-biased distribution (as a weighted distributions) is rather closely related to the Lorenz curve which is used in economics to illustrate income distributions. Connections due to Lorenz curves and with weighted distributions is discussed in this part. For special weights some remarks are noticeable. Let $U = w(X)$ and L_U be its Lorenz curve, on assuming $U \sim h(x)$, and $X \sim f(x)$, Bartoszewicz and Skolimowska (2006) proved that, let w be a monotone left continuous and increasing [decreasing] function, then $G_w(x) = L_U(F(x))$ [$G_w(x) = \overline{L}_U(\overline{F}(x))$].

Theorem 1 *Let w be a monotone left continuous and increasing [decreasing] function, then $G_w(x) = \frac{1}{E(w(X))}GL_U(F(x))$ [$G_w(x) = \frac{1}{E(w(X))}\overline{GL}_U(\overline{F}(x))$], where GL_U is the generalized Lorenz curve.*

Proof : We have, $GL_U(p) = \int_0^{H^{-1}(p)} th(t)dt$ and $H(x) = P(X \leq w^{-1}(x)) = F(w^{-1}(x))$. So, $GL_U(F(x)) = E(w(X)) \int_0^x w(t)f(t)dt$, and easy verify the results.

- For $w(x) = e^{lx}x^i[F(x)]^j[\overline{F}(x)]^k$, when

$$\frac{1}{r(x)}[l + \frac{i}{x}] + j\frac{r(x)}{\widetilde{r}(x)} > (<)k, \forall x > 0, \quad (1)$$

then $G_w(x) = L_U(F(x))$ ($G_w(x) = \bar{L}_U(\bar{F}(x))$), where $r(\cdot)$ and $\tilde{r}(\cdot)$ are hazard rate and reversed hazard rate respectively. So for $k \leq 0$ and various positive values l, i and j , $G_w(x) = L_U(F(x))$ and for weighted such as proportional hazard rate and reversed hazard rate, $G_w(x) = \bar{L}_U(\bar{F}(x))$.

- For $w(x) = [-\ln F(x)]^j [-\ln \bar{F}(x)]^k$, if $\frac{n}{F(x)} \ln \bar{F}(x) > (<) \frac{m}{\bar{F}(x)} \ln F(x)$, $\forall x > 0$, then, $w(\cdot)$ is increasing (decreasing) respectively that implies $G_w(x) = L_U(F(x))$ ($G_w(x) = \bar{L}_U(\bar{F}(x))$). Note that $m = 0$ (lower record) and $n = 0$ (upper record) are weights that are decreasing and increasing respectively. We can have the results for generalized Lorenz curve and weighted distribution also via the same arguments.
- On noting that the hazard rate of the weighted distribution and reversed hazard rate of the weighted distribution with increasing (decreasing) weights are equal to $\frac{w(x)r_F(x)}{E(w(X))} \frac{\bar{F}(x)}{\bar{L}_U(F(x))}$ ($\frac{w(x)\tilde{r}_F(x)}{E(w(X))} \frac{\bar{F}(x)}{\bar{L}_U(F(x))}$) and $\frac{w(x)r_F(x)}{E(w(X))} \frac{F(x)}{L_U(F(x))}$ ($\frac{w(x)\tilde{r}_F(x)}{E(w(X))} \frac{F(x)}{L_U(F(x))}$) respectively. We can find the increasing (decreasing) cases for the second terms in the above statements based on Lorenz curve and also, generalized Lorenz curve.

4 Lorenz Ordering for Weighted Distributions

The star shaped order was introduced by Barlow and Proschan (1975) to compare continuous life distributions. The convex order which requires the continuity of the distributions and have linked with star shaped order.

Definition 1 For $X, Y \in \mathbb{L}$ we say that X has a star shaped distribution with respect to Y and write $X \leq_* Y$, if the ratio of inverse distribution functions $F_Y^{-1}(v)/F_X^{-1}(v)$ is an increasing function in $v \in (0, 1)$. For the continuous variates $X, Y \in \mathbb{L}$ we say that X is convex ordered with respect to Y (in symbols: $X \leq_c Y$), if the function $F_Y^{-1}(F_X(x))$ is convex on the support of X .

Definition 2 The Lorenz partial order, \leq_L on \mathbb{L} is defined by

$$X \leq_L Y \iff L_X(u) \geq L_Y(u), \forall u \in [0, 1].$$

Lorenz partial order is invariant with respect to scale transformation. X is stochastically smaller than Y ($X \leq_{st} Y$) if $F(x) \geq G(x)$ for $\forall x > 0$ where F and G are distribution function of X and Y respectively.

For reasons of mathematical tractability, both the convex and the star shaped ordering can sometimes be used to verify the Lorenz ordering which they imply. The relations between the three partial orderings are presented in the next theorem.

Theorem 2 Suppose that $X, Y \in L$ are continuous. If $X \leq_c Y$, then $X \leq_* Y$ and also, $X \geq_Z Y$. Moreover, $X \leq_* Y$ implies $X \leq_L Y$.

Proof : The proof of the first part is apparent from Arnold (1987, pp. 77-78) or Barlow and Proschan (1975, p. 107), While the second part is shown in Arnold (1987, p. 78) or Moothathu (1991).

Theorem 3 Let $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be continuous function satisfying : $g(x) > 0$ for all $x > 0$, $g(x)$ is non-decreasing on $[0, \infty)$ and $\frac{g(x)}{x}$ is non-decreasing on $(0, \infty)$. If $g(X) \in L$, then, $g(X) \leq_L X$.

- For random variables X_1 and X_2 where $X_i \sim GB2(a_i, b_i, p_i, q_i)$, $i = 1, 2$ Wilfling (1990, 1996) found that $X_1 \geq_L X_2 \Rightarrow a_1 p_1 \leq a_2 p_2, a_1 q_1 \leq a_2 q_2$ where $(a_1 \leq a_2, p_1 \leq p_2, q_1 \leq q_2)$.
- A complete characterization of the Lorenz ordering with in the generalized beta family of distribution includes some of the results due to Lorenz ordering and generalized Lorenz ordering for a big class of distributions.
- The Zenga curve provide a partial ordering between random variables. The Zenga partial order, \leq_Z on \mathbb{L} is defined by $X \leq_Z Y \iff Z_X(u) \geq Z_Y(u), \forall u \in [0, 1]$.

Definition 3 We say the random variable Y is more dispersed than X in the dilation sense if $E(\phi(X - E(X))) \leq E(\phi(Y - E(Y)))$ for all convex ϕ that is denoted by $X \leq_{dil} Y$ (Ramos 2003).

Theorem 4 Let X and Y be random variables with respective finite means $E(X)$ and $E(Y)$ and let the corresponding be F and G , respectively. Then, $X \leq_{dil} Y$ if and only if $A_X(p) \geq A_Y(p), \forall p \in [0, 1]$.

- Extension of the Theorem 4, when random variable Y be a weighted distribution of X with weight w , under most of them, does not lead to a simple and nice form.
- Let X be a non-negative continuous random variable with density f , it will be increasing proportional likelihood ratio (IPLR) if $\frac{f(\lambda x)}{f(x)}$ is increasing in x for any positive constant $\lambda < 1$. It will be said decreasing proportional likelihood ratio (DPLR) if $\frac{f(\lambda x)}{f(x)}$ is decreasing in x for any positive constant $\lambda < 1$, that is defined in Ramos and Sordo Diaz (2001). IPLR and DPLR properties are sufficient conditions for the Lorenz ordering of truncated distributions and their connection with a few cases to weighted distribution is specially weights that concentrated on them and some special cases of them is interesting.
- The equilibrium distribution corresponding to F and G are F_e and G_e defined by $F_e(x) = \frac{1}{\mu_F} \int_0^x \bar{F}(t) dt$ and $G_e(x) = \frac{1}{\mu_G} \int_0^x \bar{G}(t) dt$. F is said to be more harmonic new better than used in expectation (HNBUE) than G ($F \leq_{HNBUE} G$) if and only if $\bar{F}_e(x\mu_F) \leq \bar{G}_e(x\mu_G)$. G is said to be more dispersed than F ($F \leq_{disp} G$) if $G^{-1}F(x) - x$ is increasing in x . Note that, if $\mu_F \leq \mu_G$, then, $F \leq_{HNBUE} G$ implies $F_e \leq_{disp} G_e$ which is discussed in Kochar (1989).

Theorem 5 $F \leq_{HNBUE} G$ if and only if $X \leq_L Y$.

Theorem 6 Let $X, Y \in L$, $X \leq_{st} Y$, $U = w(X) \leq_L V = w(Y)$ and w be monotone left continuous. If w is increasing (decreasing), then,

$\hat{X}_w \leq_{st} (\geq_{st}) \hat{Y}_w$ where \hat{X}_w and \hat{Y}_w are weighted version of the random variables X and Y with the weight w respectively.

Proof: We have via the assumption, $F(x) \geq G(x)$ and $L_U(p) \geq L_V(p), p \in (0, 1)$. Thus, via increasing (decreasing) w concludes that $\hat{F}_w(x) = L_U(F(x)) \geq L_V(F(x)) \geq L_V(G(x)) = \hat{G}_w(x)$ which is $\hat{X}_w \leq_{st} (\geq_{st}) \hat{Y}_w$.

5 Generalized Lorenz Order for Weighted Distributions

The Lorenz curve is only a partial order, so what does one do if two Lorenz curves intersect? The most widely used alternative to the Lorenz order is the generalized Lorenz order due to Shorrocks (1983). If F and G have equal means, the Lorenz curve of distribution F is nowhere below the Lorenz curve of distribution G , then F is preferred to G . If the generalized Lorenz curve of F is nowhere below the generalized Lorenz curve of G , then, F is preferred to G if means μ_F and μ_G are different. So we have the following definition :

Definition 4 The generalized Lorenz partial order, \leq_{GL} on \mathbb{L} is defined by

$$X \leq_{GL} Y \iff E(X)L_X(u) \leq E(Y)L_Y(u), \forall u \in [0, 1].$$

Let F and G be two distribution function of random variable X and Y respectively, then $F(x) \leq G(x), \forall x \in \mathbb{R}^+ \Rightarrow F \geq_{FSD} G$ and $\int_0^x F(t)dt \leq \int_0^x G(t)dt, \forall x \in \mathbb{R}^+ \Leftrightarrow F \geq_{SSD} G$ where FSD and SSD are "First-order stochastic dominance" and "second-order stochastic dominance" respectively.

- It can be shown that how to determine the relation between the parameters of the two generalized beta, generalized gamma families (including many income distributions as special cases) and some characterization notes in view of generalized Lorenz ordering in place of Lorenz ordering.

Definition 5 Let X and Y be two random variables with absolute Lorenz curves $A_X(t)$ and $A_Y(t)$ respectively. We say that X is smaller than Y in the second-order absolute Lorenz order if $\int_p^1 A_X(t)dt \geq \int_p^1 A_Y(t)dt, \forall p \in [0, 1]$.

- Note that the dilation order implies second-order absolute Lorenz order.
- $F \geq_{FSD} G$ if and only if $GL_F(p) - GL_G(p)$ is increasing in p . For example, let $f(x) \sim G(\alpha, \lambda), \alpha, \lambda > 0$, we know that $F \geq_L G$ if and only if $\alpha \geq \beta, \lambda \geq \nu$ and $\frac{\alpha}{\lambda} \geq \frac{\beta}{\nu}$ implies $F \geq_{SSD} G \Leftrightarrow F \geq_{GL} G$. $\lambda \leq \nu$ and $\alpha \geq \beta$ implies $F \geq_{FSD} G$.

Theorem 7 Let $X, Y \in L, X \leq_{st} Y, U = w(X) \leq_{GL} V = w(Y), E(U) \leq E(V)$ and w be monotone left continuous. If w is increasing (decreasing), then, $\hat{X}_w \leq_{st} (\geq_{st}) \hat{Y}_w$ where \hat{X}_w and \hat{Y}_w are weighted version of the random variables X and Y with the weight w respectively.

Proof: We have via the assumption, $F(x) \geq G(x)$ and $E(U)L_U(p) \geq E(V)L_V(p), p \in (0, 1)$. Thus, via increasing w concludes that $\hat{F}_w(x) = E(U)GL_U(F(x)) \geq E(V)GL_V(F(x)) \geq E(V)GL_V(G(x)) = \hat{G}_w(x)$ which is $\hat{X}_w \leq_{st} \hat{Y}_w$. $w(\cdot)$ decreasing leads to $\hat{X}_w \geq_{st} \hat{Y}_w$ via the same arguments.

6 IFRA (DFRA) and NBU (NWU) Classes Connected to Lorenz Curve and Weighted Distributions

Various representation is obtained in Bartoszewicz and Skolimowska (2006), that is connected to Lorenz curve and weighted distribution. We will mention them here with some comments. Let F be an absolutely continuous with $F(0) = 0$ and S_F be an interval. F is increasing failure rate in average (IFRA) iff $xr_F(x) \geq \int_0^x r_F(t)dt, x \in S_F$. F is decreasing failure rate in average (DFRA) iff $xr_F(x) \leq$

$\int_0^x r_F(t)dt, x \in S_F$. A distribution F is said to be NBU (NWU) if $\overline{F}(x+y) \leq (\geq) \overline{F}(x)\overline{F}(y)$ for all $x, y, x+y \in [0, \infty)$. It is well known that $IFRA \subset NBU$ and $DFRA \subset NWU$.

The following theorems obtained in Bartoszewicz and Skolimowska (2006) in view of Lorenz curve and we mentioned them via the similar arguments in view of Generalized Lorenz curve. We have the following theorems in view of generalized lorenz curve on using F is IFRA (DFRA) and

$$h(t) = \frac{w(t)\hat{F}(t)}{E(U)[1 - \hat{F}_w(t)]}, \quad (2)$$

is increasing (decreasing), then \hat{F}_w is IFRA (DFRA).

Theorem 8 Let $F(0) = 0$ and w be decreasing left continuous for which existence of the expected value of w . If $[GL_U(p)]^\alpha \leq (\geq) [E(U)]^{\alpha-1} GL_U(p^\alpha)$, for every $\alpha \in (0, 1)$ and $p \in [0, 1]$, and F is IFRA, then \hat{F}_w is IFRA (DFRA).

The next theorems are concerning the preservation of NBU and NWU classes in connection with generalized Lorenz curve on noting that if h that is defined in (2) is increasing (decreasing) and F is NBU (NWU), then \hat{F}_w is NBU (NWU).

Theorem 9 Let F be absolutely continuous, if F is IFRA (NBU) and $w(x)\overline{F}(x)$ is increasing, then \hat{F}_w is IFRA (NBU). Let w be decreasing left continuous. If F is DFRA (NWU) and $\frac{w(x)}{GL_U\overline{F}(x)}$ is decreasing, then \hat{F}_w is DFRA (NWU).

Theorem 10 Let $F(0) = 0$ and F be NBU (NWU). Let w be increasing (decreasing) left continuous for which existence of the expected value of w . If $GL_U(pq) \leq (\geq) [E(U)]GL_U(p)GL_U(q)$, for every $p, q \in [0, 1]$. Then \hat{F}_w is NBU (NWU).

7 Conclusions

In this paper, notes due to Lorenz and generalized Lorenz curve is discussed in view of weighted distributions and their order is derived in view of Bartoszewicz and Skolimowska (2006) based on weighted version. Properties of these two order found via two general form weights that some special cases of them are very famous and important at least in reliability. Characterization results is also obtained via the above arguments.

References

- [1] Arnold, B.C. (1987). Majorization and the Lorenz Order. Lecture Notes in Statistics 43, Springer-Verlag, Berlin - New York.
- [2] Atkinson, A. B. (1970). On the measurement of inequality. *J. of Economics Theory*, 2, 244-263.
- [3] Barlow, R.E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing. Holt, Rinehart and Winston, New York.
- [4] Bartoszewicz, J. and Skolimoska, M. (2006). Preservation of classes of life distributions under weighting. *Stat. and Prob. Letters*, **76**, 587-596.
- [5] Jain, K., Singh, H. and Bagai, I. (1989). Relation for reliability measures of weighted distributions. *Comm. Statistics, Theory and Methods*, **18**, 4393-4412.
- [6] Kakwani, N. (1984). Welfare ranking of income distributions. *Advances in Econometrics*, 3, 191-213.
- [7] Kleiber, C. and Kotz, S. (2003). *Statistical size Distribution in Economics and Actuarial Sciences*. John Wiley.
- [8] Kochar, S. C. (1989). On extension of DMRL and related partial ordering of life distribution. *Comm. Statistics, Stochastics Models*, **5**, 235-245.

- [9] Lorenz, M. O. (1905). Methods for measuring concentration of wealth. *Journal of the American Statistical Association* 9, 209-219.
- [10] Moothathu, T.S.K. (1991). On the sufficient condition for two non-intersecting Lorenz curves. *The Indian Journal of statistics*. Volume, 53. series, B. pt, 2. pp, 268-274.
- [11] Ramos, H. M. and Sordo, M. A. (2001). The proportional likelihood ratio order and applications. *QUESTIO*, **61**, 123-131.
- [12] Ramos, H. M. and Sordo, M. A. (2003). Dispersion measures and dispersive orderings. *Stat. and Prob. Letters*, **25**, 2, 211-223.
- [13] Rao, C. R. (1965) . On discrete distributions arising out of methods of ascertainment, in *Classical and Contagious Discrete Distributions*, G. Patil Ed., Pergamon Press and Statistical Publishing Society, Calcutta, 320-332.
- [14] Sarabia, J. Castillo, E. and Slottje, D. (1999). An ordered family of Lorenz curves. *Journal of Econometrics*, 91. 43-60.
- [15] Shorrocks, A. F. (1983). Ranking income distributions, *Economica* 50, 3-17.
- [16] Thistle, P. D. (1989). Duality between generalized Lorenz curves and distribution functions. *Economic Studies Quarterly*, 40, 183-187.
- [17] Thompson, W. A., Jr. (1976). Fishermans luck. *Biometrics*, 32, 265-271.
- [18] Wilfling, B.(1996). A sufficient condition for Lorenz ordering. *The Indian Journal of statistics*. Volume, 58. series, B. pt, 1. pp, 62-69.
- [19] Wilfling, B. (1990). Concepts of Lorenz-ordering with a special regard to the Lorenz ordering of order statistics. Diploma Thesis, Fak. Statist., Univ. Dortmund (Germany).