Pergamon

Appl. Math. Lett. Vol. 7, No. 5, pp. 93–96, 1994
Elsevier Science Ltd. Printed in Great Britain

0893-9659(94)00079-4

Laplace Transform Pairs of N-Dimensions

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(Received October 1993; accepted November 1993)

Abstract—The main theme of the present paper is to establish a formula for calculating the Inverse Laplace transformation of functions of the form $p_1(\overline{s^{1/2}})/p_n(\overline{s^{1/2}})\, F[(p_1(\overline{s^{1/2}}))^2]$. This result is used to obtain the Inverse Laplace transforms of generalized hypergeometric and Lommel functions of n variables.

Keywords—Multidimensional Laplace transformations.

1. INTRODUCTION AND NOTATION

In spite of the numerous applications of one-dimensional Laplace transformation, the idea naturally arose of generalizing the transform functions of n variables. However, the determination of the original function $f(\bar{x})$ from the following complex inversion formula is often a great difficulty.

$$\mathcal{L}_n^{-1}\{F(\bar{s});\bar{x}\} = f(\bar{x}) = (2\pi i)^{-n} \int_{(\bar{\alpha})} \exp(\bar{s}\cdot\bar{x})F(\bar{s})\,d\bar{s}, \qquad \bar{\alpha}\in\mathbb{R}_+^n,\ \bar{\alpha}>\bar{\alpha},$$

where $\bar{x} = (x_1, x_2, \dots, x_n), \bar{s} = (s_1, s_2, \dots, s_n)\bar{\alpha} \in \mathbb{R}_+^n$, and $\mathbb{R}_+^n = \{\bar{x} : \bar{x} \in \mathbb{R}^n, \bar{x} > 0\}$.

In this paper, we obtained the inverse Laplace transformations of functions of the form $p_1(\overline{s^{1/2}})/p_n(\overline{s^{1/2}})F[(p_1(\overline{s^{1/2}}))^2]$, where $p_1(\overline{s^{1/2}}) = s_1^{1/2} + s_2^{1/2} + \cdots + s_n^{1/2}$ and $p_n(\overline{s^{1/2}}) = s_1^{1/2}\cdot s_2^{1/2}\dots s_n^{1/2}$. In the present work, the original space Ω (or class Ω) is taken to consist of all complex valued functions f that is sectionally continuous for $x \ge 0$ and of exponential order as $x \to \infty$.

2. THE MAIN THEOREM

THEOREM. *Assume that $f(x^2)$, $s^{-1/2}\phi(1/x)$ and $x^{-3/2}\xi(1/x^2)$ are functions of class Ω. Let*

(i) $\mathcal{L}\{f(x);s\} = \phi(s)$,
(ii) $\mathcal{L}\left\{x^{-1/2}\phi(1/x);s\right\} = \xi(s)$,
(iii) $\mathcal{L}\left\{x^{-3/2}\xi\left(1/x^2\right);s\right\} = \theta(s)$,
(iv) $\mathcal{L}\left\{xf\left(x^2\right);s\right\} = H(s)$,

where $x^{-1/2}\exp(-sx - u/x)f(u)$ and $x^{-1/2}\exp\left(-sx - 2u^{1/2}/x\right)f(u)$ belong to $L_1[(0,\infty)\times(0,\infty)]$. Then,

$$\mathcal{L}_n^{-1}\left\{\frac{p_1(\overline{s^{1/2}})}{p_n(\overline{s^{1/2}})}\,\theta\left[\left((\overline{s^{1/2}})\right)^2\right];\bar{x}\right\} = \frac{2}{\pi^{(n-2)/2}\,p_n(\overline{x^{1/2}})}H\left[2p_1(\overline{x^{-1}})\right]. \tag{2.1}$$

It is assumed that the integrals involved exist for $n = 2, 3, \dots, N$.

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PROOF. First, we apply the definition of one-dimensional Laplace transform to obtain

$$\int_0^\infty x^{-1/2}\exp(-sx)\phi\left(\frac{1}{x}\right)dx = \int_0^\infty \left[\int_0^\infty x^{-1/2}\exp\left(-sx-\frac{u}{x}\right)f(u)\,du\right]dx. \tag{2.2}$$

The integrand $x^{-1/2}\exp(-sx-u/x)f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$, so, by Fubini's Theorem, interchanging the order of the integral on the right side of (2.2) is permissible. By using (ii) on the left side and interchanging the order of integration on the right side of (2.2), we obtain

$$\xi(s) = \int_0^\infty f(u)\left[\int_0^\infty x^{-1/2}\exp\left(-sx-\frac{u}{x}\right)dx\right]du, \quad \text{where } \operatorname{Re} s > c_1 \text{ and } c_1 \text{ is a constant.} \tag{2.3}$$

From the tables by Roberts and Kaufman [1], (2.3) reads

$$\xi(s) = \left(\frac{\pi}{s}\right)^{1/2}\int_0^\infty f(u)\exp\left(-2u^{1/2}s^{1/2}\right)du. \tag{2.4}$$

Plugging (2.4) in (iii), we arrive at

$$\theta(s) = \pi^{1/2}\int_0^\infty\left[\int_0^\infty x^{-1/2}f(u)\exp\left(-sx-\frac{2u^{1/2}}{x}\right)du\right]dx, \quad \operatorname{Re} s > c_2 \text{ and } c_2 \text{ is a constant.} \tag{2.5}$$

By the hypothesis, $x^{-1/2}f(u)\exp\left(-sx-(2u^{1/2}/x)\right)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$. Using Fubini's Theorem, we interchange the order of integration on the right side of (2.5). Next, we use a well-known result in Roberts and Kaufman [1] on the right-hand side of the resulting relation to obtain

$$\theta(s) = \pi\int_0^\infty s^{-1/2}f(u)\exp\left(-2^{3/2}u^{1/4}s^{1/2}\right)du, \quad \operatorname{Re} s > c_2. \tag{2.6}$$

By substituting $u = v^2$ in (2.6), and replacing s with $[p_1(\overline{s^{1/2}})]^2$ and next multiplying both sides of the results relation by $p_n(\overline{s^{1/2}})$, we arrive at

$$p_1(\overline{s^{1/2}})p_n(\overline{s^{1/2}})\theta\left[\left(p_1(\overline{s^{1/2}})\right)^2\right] = 2\pi\int_0^\infty vf\left(v^2\right)p_n(\overline{s^{1/2}})\exp\left(-2^{3/2}v^{1/2}p_1(\overline{s^{1/2}})\right)dv. \tag{2.7}$$

Using the following known result from Ditkin and Prudnikov [2]:

$$s_i^{1/2}\exp\left(-as_i^{1/2}\right) \doteqdot (\pi x_i)^{-1/2}\exp\left(-\frac{a^2}{4x_i}\right), \qquad \text{for } i = 1,2,\dots,n.$$

Equation (2.7) can be rewritten as

$$p_1(\overline{s^{1/2}})p_n(\overline{s^{1/2}})\theta\left[\left(p_1(\overline{s^{1/2}})\right)^2\right] \underset{n}{\doteqdot} \frac{2}{\pi^{(n-2)/2}p_n(\overline{x^{1/2}})}\int_0^\infty vf\left(v^2\right)\exp\left(-2vp_1(\overline{x^{-1}})\right)dv. \tag{2.8}$$

Using (iv) in (2.8), we obtain

$$p_1(\overline{s^{1/2}})p_n(\overline{s^{1/2}})\theta\left[\left(p_1(\overline{s^{1/2}})\right)^2\right] \underset{n}{\doteqdot} \frac{2}{\pi^{(n-2)/2}p_n(\overline{x^{1/2}})}H\left[2p_1(\overline{x^{-1}})\right].$$

Therefore,

$$\mathcal{L}_n^{-1}\left\{\frac{p_1(\overline{s^{1/2}})}{p_n(\overline{s^{1/2}})}\theta\left[\left(p_1(\overline{s^{1/2}})\right)^2\right];\bar{x}\right\} = \frac{2}{\pi^{(n-2)/2}p_n(\overline{x^{1/2}})}H\left[2p_1(\overline{x^{-1}})\right].$$

3. APPLICATIONS

EXAMPLE 3.1. Suppose that $f(x) = (\pi x)^{-1/2}\cos\left[2/a\, x^{1/2}\right]$. Then,

$$\phi(s) = s^{-1/2}\exp\left(-\frac{1}{a^2 s}\right), \qquad \operatorname{Re} s > 0.$$

$$\xi(s) = \frac{1}{s+1/a^2}, \qquad \operatorname{Re} s > -\frac{1}{a^2}.$$

$$\theta(s) = \frac{a^2 p^{1/2}}{2} s^{1/2} \underset{-1,1/2}{S}(as), \qquad \operatorname{Re} a > 0,\ \operatorname{Re} s > 0.$$

$$H(s) = \frac{\pi s}{s^2 + \left(\frac{2}{a}\right)^2}, \qquad \operatorname{Re} s > 2\left|\operatorname{Im}\frac{1}{a}\right|.$$

Using (2.1), we arrive at

$$\mathcal{L}_n^{-1}\left\{\frac{p_1^2(\overline{s^{1/2}})}{p_n(\overline{s^{1/2}})} \underset{-1,1/2}{S}\left[a\left(p_1(\overline{s^{1/2}})\right)^2\right]; \bar{x}\right\} = \frac{2p_1(\overline{x^{-1}})}{\pi^{(n-3)/2} p_n(\overline{x^{1/2}})\left[a^2 p_1^2(\overline{x^{-1}}) + 1\right]},$$

where $\operatorname{Re} a > 0$, $\operatorname{Re}\left[p_1(\overline{s^{1/2}})\right] > 0$, $n = 2, 3, \dots, N$.

EXAMPLE 3.2. Let $f(x) = x^{-\alpha} G_{h,k}^{m,n}\left(x\middle|{}^{a_1,\dots,a_h}_{b_1,\dots,b_k}\right)$, where $h + k < 2(m+n)$, $\operatorname{Re}\alpha > \operatorname{Re} b_j + 1$, $j = 1, \dots, m$. Then,

$$\phi(s) = s^{\alpha-1} G_{h+1,k}^{m,n+1}\left(\frac{1}{s}\middle|{}^{\alpha,a_1,\dots,a_h}_{b_1,\dots,b_k}\right), \qquad |\arg s| < \left(m + n - \frac{1}{2}h - \frac{1}{2}k\right)\pi.$$

The same formula is valid if $h < k$ (or $h = k$ and $\operatorname{Re}, s > 1$) and $\operatorname{Re}\alpha < \operatorname{Re} b_j + 1$, $j = 1, 2, \dots, m$.

$$\xi(s) = s^{\alpha-(3/2)} G_{h+2,k}^{m,n+2}\left(\frac{1}{s}\middle|{}^{(\alpha-1/2),\alpha,a_1,\dots,a_h}_{b_1,\dots,b_k}\right), \qquad |\arg s| < \left(m + n - \frac{1}{2}h - \frac{1}{2}k + \frac{1}{2}\right)\pi.$$

$$\theta(s) = \pi^{-1/2} 2^{-\alpha} s^{\alpha-1} G_{h+4,k}^{m,n+4}\left(\frac{4}{s^2}\middle|{}^{(\alpha-1/2),\alpha,(\alpha/2),(\alpha+1)/2,a_1,\dots,a_h}_{b_1,\dots,b_k}\right),$$

where $\operatorname{Re}(1-\alpha) + 2\min_j \operatorname{Re} b_j > 0 (j = 1, 2, \dots, k)$, $\operatorname{Re} s > 0$.

$$H(s) = \pi^{-1/2} 2^{(1-2\alpha)} s^{(2\alpha-2)} G_{h+2,k}^{m,n+2}\left(\frac{4}{s^2}\middle|{}^{(2\alpha-1)/2,\alpha,a_1,\dots,a_h}_{b_1,\dots,b_k}\right),$$

where $\operatorname{Re}(2-2\alpha) + 2\min_j \operatorname{Re} b_j > 0 (j = 1, 2, \dots, k), \operatorname{Re} s > 0$. Now we apply the result of this Theorem, to obtain

$$\mathcal{L}_n^{-1}\left\{\frac{\left[p_1(\overline{s^{1/2}})\right]^{2\alpha-1}}{p_n(\overline{s^{1/2}})} G_{h+4,k}^{m,n+4}\left(\frac{4}{p_1^4(\overline{s^{1/2}})}\middle|{}^{(\alpha-1/2),\alpha,\alpha/2,(\alpha+1)/2,\,a_1,\dots,a_h}_{b_1,\dots,b_k}\right); \bar{x}\right\}$$

$$= \frac{2^\alpha}{\pi^{(n-2)/2} p_n(\overline{x^{1/2}})} G_{h+2,k}^{m,n+2}\left(\frac{1}{p_1^2(\overline{x^{-1}})}\middle|{}^{(\alpha-1/2),\alpha,a_1\dots,a_h}_{b_1,\dots,b_k}\right),$$

where

$$\operatorname{Re}\alpha < 1 + 2\min_j \operatorname{Re} b_j (j = 1, 2, \dots, k), \operatorname{Re}\left[p_1(\overline{s^{1/2}})\right] > 0.$$

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