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Linear and Non-linear Analyses of Functionally Graded Plates Using Generalized Differential Quadrature Method

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ABSTRACT

Linear and non-linear static analyses of moderately thick functionally graded (FG) rectangular plates with different boundary conditions are presented using generalized differential quadrature (GDQ) method. The modulus of elasticity of the plates is assumed to vary according to a power law distribution in terms of the volume fractions of the constituents. Non-linear system of equations is based on the first-order shear deformation theory and von Kármán-type of geometric non-linearity. To derive linear system of equations, non-linear terms are omitted in former equations. The governing system of equations includes a system of thirteen partial differential equations in terms of unknown displacements, forces and moments. Presence of all plate variables in the governing equations provides a simple procedure to satisfy different boundary conditions. Application of the GDQ technique to the governing equations, solution domain and boundary conditions leads to a system of non-linear algebraic equations. The Newton–Raphson iterative scheme is then employed to solve the resulting system of non-linear equations. Illustrative examples are presented to demonstrate accuracy and rapid convergence of the presented GDQ technique. Accuracy of the results for both displacement and stress components are verified with comparing the present results with those of analytical and finite element methods. It is found that the theory can predict accurately the displacement and stress components even for small number of grid points.

1. INTRODUCTION

To avoid the material mismatch, a special material named ‘‘functionally graded material’’ (FGM) was proposed by a group of material scientists in Japan [1], in which the material properties are varied smoothly and continuously from one surface to the other. The gradation of the material properties through the thickness eliminates jumps or abrupt changes in the stress and displacement distributions. The FGM is suitable for various applications, such as thermal coatings of barrier for ceramic engines, optical thin layers, biomaterial electronics, nuclear fusions, gas turbines, etc.

In reality, many plate structures are subjected to high load levels that may undergo large

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deflections. The effect of this large deflection is to stretch the middle plane of the plate inducing membrane stresses. By this membrane action, the load carrying capacity of the plate is increased to a large extent. For plates of this kind, the governing differential equations become non-linear. The non-linearity of the governing equations may be due to either material non-linearity or geometric non-linearity. In this paper only geometric non-linearity will be considered. This non-linearity is due to the fact that the strain displacement relations are non-linear. A considerable amount of literature exists on the non-linear analysis of rectangular plates. Zenkour [2] used generalized shear deformation theory for bending analysis of functionally graded plates. Woo and Meguid [3] provided an analytical solution for the large deflections of functionally graded plates and shallow shells under transverse mechanical loads and a temperature field using Fourier series. Yang and Shen [4] studied non-linear bending analysis of shear deformable functionally graded plates subjected to thermo-mechanical loads. They also investigated the large deflection and postbuckling of functionally graded rectangular plates under transverse and in-plane loads by using a semi-analytical approach [5]. Tsung and Shukla [6] provided an explicit solution for the non-linear static and dynamic responses of a functionally graded rectangular plate using the quadratic extrapolation technique for linearization, finite double Chebyshev series for spatial discretization of the variables and Houbolt time marching scheme for temporal discretization. Reddy [7] developed Navier's solutions for rectangular plates and finite element models to study the non-linear dynamic response of FG plates using the higher-order shear deformation plate theory (HSDPT). Based on Reddy's HSDPT, Shen [8] also studied the non-linear bending of a simply-supported functionally graded rectangular plate subjected to mechanical and thermal loads. Navazi et al. [9] developed an analytical solution for non-linear cylindrical bending of a functionally graded plate. They showed that the linear plate theory is inadequate for analysis of functionally graded plates even in small deflection ranges. GhannadPour and Alinia [10] obtained an analytical solution for large deflection of rectangular functionally graded plates under pressure loads by minimization of the total potential energy.

It is well-known that analytical methods are only applicable to particular problems such as non-linear bending of FG plates with at least two opposite sides simply supported. Thus, numerical techniques, as alternatives to analytical approaches, have been developed to obtain solutions for FG plates subjected to different types of boundary conditions. Among these numerical methods, one can refer to differential quadrature (DQ) [11] and generalized differential quadrature (GDQ) [12-13]. The DQ and GDQ techniques were presented by Bellman et al. [11] and Shu [12] as efficient procedures to obtain solutions of partial differential equations with relatively small number of grid points and less computational effort. The main advantage of the GDQ over DQ is its ease of the computation of weighting coefficients without any restriction on the choice of grid points.

In this paper, generalized differential quadrature method is employed to obtain solutions for linear and non-linear bending analyses of moderately thick FG plates. For the assessment of the accuracy and convergence of the method, the present results are compared with those of other investigators. It is found that the GDQ method predicts accurately both the displacement and stress components.

2. GOVERNING EQUATIONS

Consider a rectangular plate of sides A and B and thickness h, shown in Figure 1. According to the first-order shear deformation theory, the displacement field is expressed as [14]:

$$\begin{aligned} U(x,y,z) &= u(x,y) + z\phi_x(x,y) \\ V(x,y,z) &= v(x,y) + z\phi_y(x,y) \\ W(x,y,z) &= w(x,y) \end{aligned} \tag{1}$$

where u,v and w denote the displacements in x,y and z directions and ϕ_x and ϕ_y are rotations of xz and yz planes of mid-plane, respectively. The von Kármán non-linear strain–displacement relations can be expressed as follows [14]:

$$\{\varepsilon\} = \{\varepsilon^0\} + z\{\kappa\}$$

$$\varepsilon_x^0 = \frac{\partial u}{\partial x} + \underline{\frac{1}{2}(\frac{\partial w}{\partial x})^2}, \quad \varepsilon_y^0 = \frac{\partial v}{\partial y} + \underline{\frac{1}{2}(\frac{\partial w}{\partial y})^2}, \quad \varepsilon_{xy}^0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \underline{(\frac{\partial w}{\partial x})(\frac{\partial w}{\partial y})}, \quad \varepsilon_{xz}^0 = \frac{\partial w}{\partial x} + \phi_x, \quad \varepsilon_{yz}^0 = \frac{\partial w}{\partial y} + \phi_y$$

$$\kappa_x = \frac{\partial \phi_x}{\partial x}, \quad \kappa_y = \frac{\partial \phi_y}{\partial y}, \quad \kappa_{xy} = \frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y} \tag{2}$$

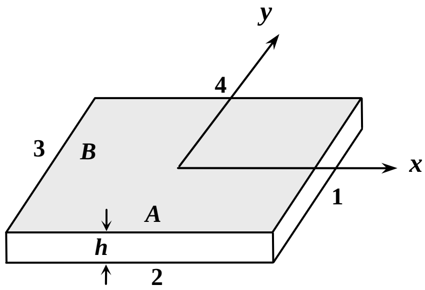


Figure 1: The geometry of plate

Underlined terms in Equations (2) are non-linear terms which are omitted for linear analysis. Integrating the relevant stress components over the thickness of the plate, the in-plane stress resultants (N_x, N_y, N_{xy}), moment resultants (M_x, M_y, M_{xy}) and transverse shear stress resultants (Q_x, Q_y), are obtained. Consequently, constitutive equations are stated as follows [14]:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix}, \quad \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = K_s \begin{bmatrix} A_{55} & A_{54} \\ A_{45} & A_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xz}^0 \\ \varepsilon_{yz}^0 \end{Bmatrix} \tag{3}$$

where K_s is the shear correction factor and in all presented results $K_s=5/6$ and

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h./2}^{h/2} Q_{ij}(1, z, z^2)dz \qquad i, j = 1,2,4,5,6$$

For FGMs we have:

$$Q_{11} = Q_{22} = \frac{E(z)}{1-\nu^2}, \quad Q_{12} = Q_{21} = \frac{\nu E(z)}{1-\nu^2}, \quad Q_{44} = Q_{55} = Q_{66} = \frac{E(z)}{2(1+\nu)}, \quad Q_{16} = Q_{61} = Q_{26} = Q_{62} = Q_{45} = Q_{54} = 0$$

$$E(z) = (E_c - E_m)\left(\frac{z}{h} + \frac{1}{2}\right)^n + E_m \tag{4}$$

where c and m denote ceramic and metal, respectively.

Equations of equilibrium can be derived using variational principle which, for the sake of brevity, is not presented here (see, for example [14]). Eight constitutive equations and five equations of equilibrium are thirteen partial differential equations governing the non-linear bending of FG plates.

Boundary conditions

Tow different boundary conditions are considered at each edge of the plate as [14]:

1- Clamped support (C):

$$u = v = w = \phi_x = \phi_y = 0 \qquad \text{at all edges} \tag{5}$$

2- Simple support (S):

$$u = v = w = \phi_y = M_x = 0 \qquad \text{at } x\text{=constant} \tag{6a}$$

$$u = v = w = \phi_x = M_y = 0 \qquad \text{at } y\text{=constant} \tag{6b}$$

Application of GDQ

In order to use GDQ technique the plate is divided into $n_x \times n_y$ grid points where (x,y) of grid points is zeros of the well-known Chebyshev polynomials [12].

$$x_i = \frac{A}{2}[1-\cos(\frac{i-1}{N_r-1}\pi)] \quad i=1,2,...,N_x, \qquad y_j = \frac{B}{2}[1-\cos(\frac{j-1}{N_y-1}\pi)] \quad j=1,2,...,N_y \tag{7}$$

According to the GDQ method, the governing equations can be re-written in discretized form. For example, the first equation in (3) at a sample grid point (x_i, y_j) can be written as [12]:

$$
\begin{aligned}
&-N_x\left(x_i,y_j\right)+A_{11}\left(\sum_{k=1}^{n_x}C^{(1)}_{ik}u\left(x_k,y_j\right)+\left(\frac{1}{2}\right)\left(\sum_{k=1}^{n_x}C^{(1)}_{ik}w\left(x_k,y_j\right)\right)^2\right)+\\
&A_{12}\left(\sum_{k=1}^{n_y}\bar{C}^{(1)}_{jk}v\left(x_i,y_k\right)+\left(\frac{1}{2}\right)\left(\sum_{k=1}^{n_y}\bar{C}^{(1)}_{jk}w\left(x_i,y_k\right)\right)^2\right)+B_{11}\sum_{k=1}^{n_x}C^{(1)}_{ik}\phi_x\left(x_k,y_j\right)+B_{12}\sum_{k=1}^{n_y}\bar{C}^{(1)}_{jk}\phi_y\left(x_i,y_k\right)=0
\end{aligned}
\tag{13}
$$

where (x_i,y_j) is a grid point inside the plate with i = 2,3,..., n_x–1 and j = 2,3,..., n_y–1. $C^{(1)}_{ij}$ and $\bar{C}^{(1)}_{ij}$ are weighting coefficients for first order partial derivatives [12]. Following the procedure explained above leads to a system of 13(n_x×n_y) non-linear algebraic equations with the same number of unknowns. It should be noted that applying boundary conditions in Equations (5) and (6) to the obtained algebraic equations, five of the thirteen unknown parameters at each boundary node will vanish. At last an incremental-iterative method should be used to solve the resulting non-linear system of equations. In the present analysis, the solution algorithms are based on the Newton–Raphson method.

3. RESULTS AND DISCUSSION

To demonstrate the efficiency and accuracy of the present method, GDQ results for both linear and non-linear bending of FG rectangular plates are validated with those of other numerical and analytical methods. In all examples the plate is considered to be constructed from Aluminum-Alumina FGM (see [2] for material properties) and transverse load (Q) is considered to be uniform.

First example is linear bending of fully simply-supported FG plate. Figure 2(a) shows normalized deflection of SSSS rectangular FG plate versus *A/B* in comparison with analytical results of Zenkour [2]. Normalized deflection of SSSS square FG plate versus *A/h* is also shown in Figure 2(b).

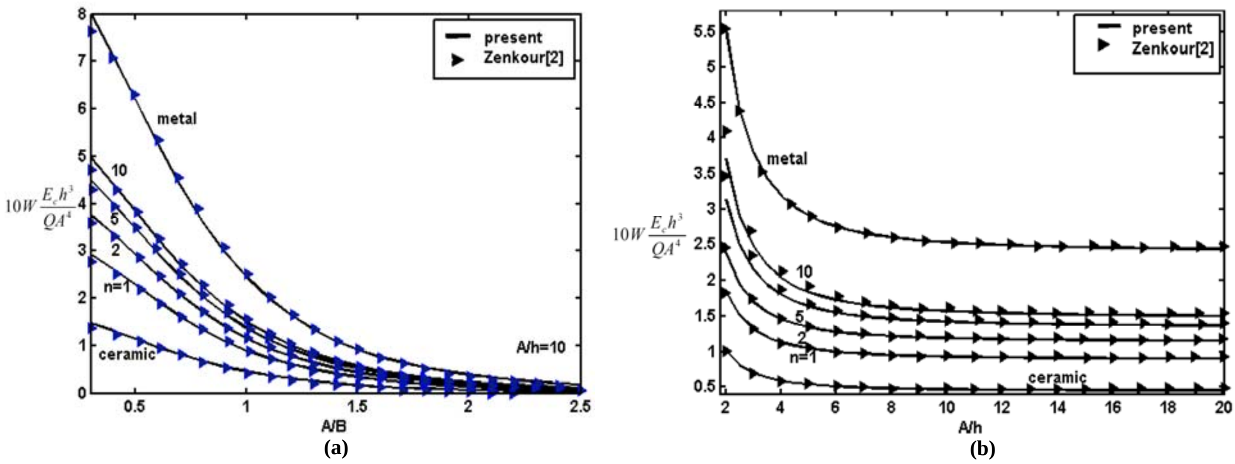
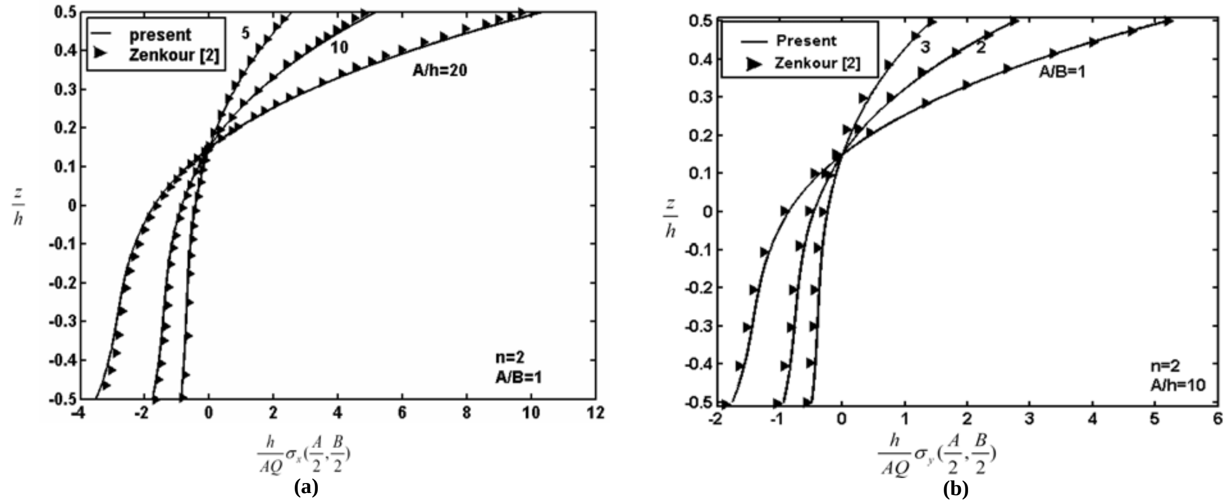


Figure2: (a) Normalized deflection of SSSS FG rectangular plate versus *A/B*
(b) Normalized deflection of SSSS FG square plate versus *A/h* (linear analysis).

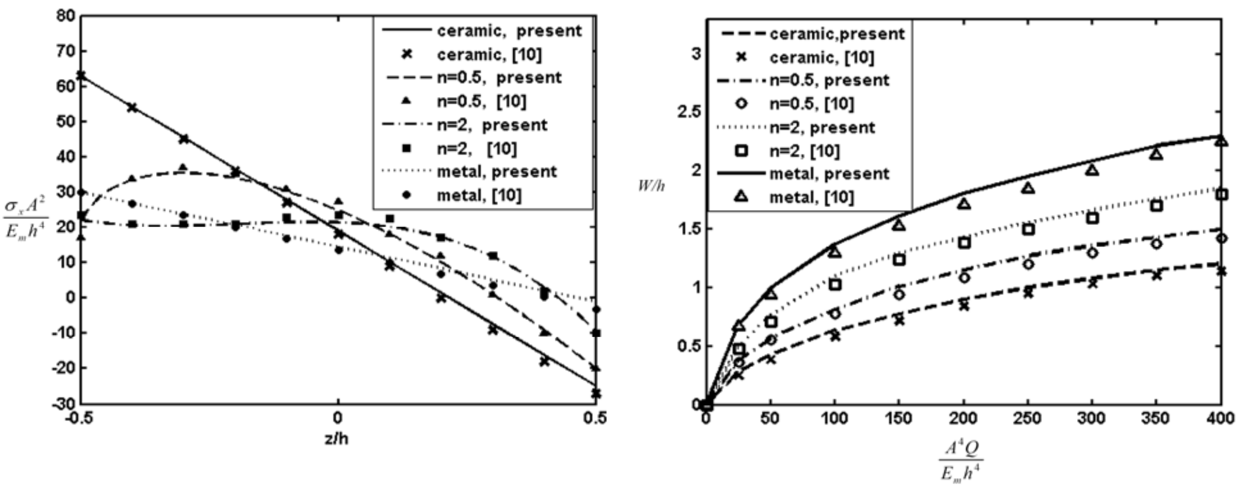
The system of equations which is used here guarantees the same order of accuracy for predictions of various stress and displacement components. In order to prove the idea variations of in-plane longitudinal and lateral normal stresses through the thickness of SSSS FG plate are demonstrated in Figures 3(a) and 3(b), respectively.

Second example regards to non-linear analysis for FG plates. Figure 4(a) shows through the thickness variation of normalized in-plane longitudinal stress of a fully simply-supported FG square plate of side A=200 mm and thickness h=10 mm. Normalized central deflection of this plate versus load is also shown in Figure 4(b). Included in these figures are also analytical results of GhannadPour and Alinia [10]. It is seen that the GDQ results are in good agreement with those obtained by analytical solution. As noticed before, presence of all variables in system of equations provides a



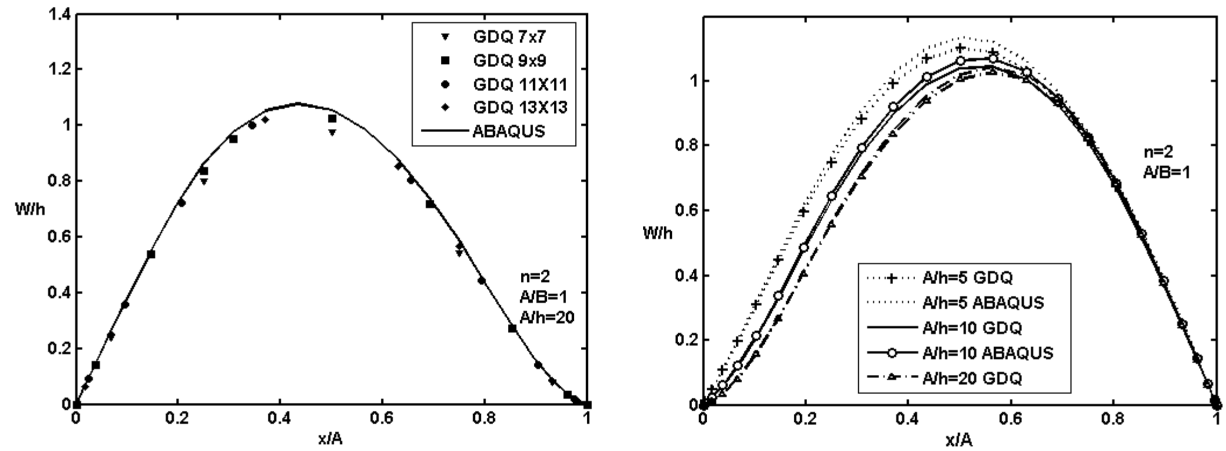
(a) (b)

Figure3: (a) Variation of in-plane longitudinal stress of SSSS FG square plate (b) Variation of in-plane normal stress of SSSS FG rectangular plate through the thickness (linear analysis).



(a) (b)

Figure4: (a) Variation of in-plane longitudinal stress of SSSS FG square plate through the thickness (b) Normalized central deflection of SSSS FG square plate versus load (non-linear analysis).



(a) (b)

Figure5: (a) Variation of normalized deflection of SSCS FG square plate at $y/B = 0.5$ (b) Variation of normalized deflection of CCSS FG square plate at $y/B = 0.5$ (non-linear analysis).

simple procedure to apply different boundary conditions. To define different boundary conditions the edges of the plate are numbered from 1 to 4 as shown in Figure 1. Last examples of this study regards to non-linear analysis of SSCS and CCSS plates. Variation of normalized deflection of SSCS and CCSS square FG plates at $y/B = 0.5$ are shown in Figures 5(a) and 5(b), respectively. In this figures $q = QA^4 / E_m h^4 = 200$ and the results are validated with those obtained by ABAQUS finite element package. Good accuracy of the GDQ method is a noticeable point of all examples.

4. CONCLUSIONS

The Differential Quadrature (DQ) method is used to obtain numerical solution for linear/non-linear bending of FG plates subjected to uniformly distributed load and different boundary conditions. Comparisons of the results with those available in the literature show good agreement for both displacements and stress components. It is shown that the method which is used in this study provides a simple procedure to apply different boundary conditions. The results are also revealed that the method is efficient and accurate and therefore, could be used for more complicated problems.

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