Solution of the Vector Wave Equation for Dielectric Rod Waveguides Using the Modified Fourier Decomposition Method

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Abstract-*Fundamental modes of dielectric rod waveguides of arbitrary cross-section can be determined by mapping from the infinite domain to a unit square and expanding the solution vectors in the transformed region by sine functions. The transformation is such that radiation in the x-y plane is permitted. The method is applied to the problem of rectangular dielectric waveguide.*

I. INTRODUCTION

Dielectric waveguides have been found many applications in optical systems. These optical building blocks have applications in millimeter and microwave frequencies. For example in microwave region, a segment of a dielectric waveguide could be used as dielectric resonator. The main advantage of these dielectric composed devices is their low loss at high frequencies. Complete knowledge of their dispersion properties and the fields' distributions make them be exploited more easily in microwave devices.

The problem of dielectric waveguide with arbitrary cross section is not solved analytically. Only the circular dielectric waveguides could have analytical solutions due to the correspondence between the shape and the symmetries needed for cylindrical problems. One of such arbitrary shapes is the rectangular waveguide which is widely used in optical region. The common property of such waveguides is the proximity of the refraction index of the core and cladding. This enables the use of scalar approximations as was used by many authors. When this difference is considerable, the approximate vector solution methods are used like the one proposed by Marcatili [1] and Sharma et al [2]. These solutions have a region of validity and are not of high validity near cut-off. There are also numerical methods that tend to reach to the exact solution, but some of them do not yield analytical expressions (finite element [3], finite difference, method of lines), some of them consider some walls in the far regions and do not permit radiations [4] and some of them get more complex (size of the matrices enlarges) when the geometry of the problem gets more complex, such as the method proposed by Goell [5]. In his method, the fields in every homogenous region are expanded in terms of circular harmonics and point matching is used to complete the solution. This method yields the most exact solutions when the geometry is composed of step index parts.

In this paper we propose a simple, expandable method for obtaining full wave solutions of the isolated arbitrary shaped waveguides. The method uses a mapping technique similar to the one implied in [6] and expands the field components mentioned in [4]. The mapping technique will be introduced in section II and the solution in the mapped domain will be discussed. Section III focuses on numerical results and the comparison between the results obtained by different methods.

II. MAPPING TECHNIQUE

To avoid the continuous expansions which occur in infinite domains, a mapping can be used to map the infinite 2-D domain to a finite region. This method was used in [6]. In the translated domain, the governing equations have variable coefficients, thus modal expansion will be as difficult as the first problem. As was used by [6], in this paper we use basis functions of the finite region that make a complete set for square integrable functions.

The solutions of the Helmoholtz equation have decaying behavior of $1/\sqrt{r}$. Mapping of the r component will result in the correct decaying behavior for all directions. The solutions will be expanded in the mapped domain by the circular harmonics having zero value on the wall of the region. The decaying behavior of the Bessel functions near its zeros is of the form $O(\rho - z_{nm})$, where z_{nm} is the m'th zero of the Bessel function of the first kind and order n. A transformation will be used to map from r-ϕ plane to ρ-ϕ plane ($\rho<1$):

$$r = \alpha \frac{\sin\left(\frac{\pi\rho}{2}\right)}{\cos^2\left(\frac{\pi\rho}{2}\right)} \tag{1}$$

Here α is a scaling parameter that will speed up the convergence if well chosen.

To solve the problems having rectangular shapes, it is easier to use Cartesian components, E_x and E_y and as it was shown by Marcuse [4] that in the problems having no geometrical variation in z-direction (like waveguides), the transverse components of electric or magnetic fields make a complete knowledge about the problem and the other four components can be found using derivatives of the two components (E_x and E_y, or H_x and H_y). The electric vector wave equation in the source free region is of the form of

$$\nabla^2\mathbf{E} + k_0^{\,2}\varepsilon_r(x,y)\mathbf{E} + \nabla\left(\mathbf{E}.\nabla \ln\varepsilon_r(x,y)\right) = 0 \tag{2}$$

Assuming $e^{-j\beta z}$ variations along the z-direction, for transverse electric fields there are two coupled equations:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_0^{\,2}\varepsilon_r(x,y)-\beta^2\right)E_x+\frac{\partial}{\partial x}\left(E_x\frac{\partial}{\partial x}\ln\varepsilon_r(x,y)+E_y\frac{\partial}{\partial y}\ln\varepsilon_r(x,y)\right)=0$$
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_0^{\,2}\varepsilon_r(x,y)-\beta^2\right)E_y+\frac{\partial}{\partial y}\left(E_x\frac{\partial}{\partial x}\ln\varepsilon_r(x,y)+E_y\frac{\partial}{\partial y}\ln\varepsilon_r(x,y)\right)=0 \tag{3}$$

These two will be enough for the whole solution. E_x and E_y must be of decaying behavior of at least $1/\sqrt{r}$ to make E_θ have the same behavior. Expanding them by circular harmonics, we will have

$$E_x = \sum_{n=1}^{N}\sum_{m=1}^{M} a_{mn} J_n(z_{nm}\rho)e^{jn\phi} \tag{4}$$
$$E_y = \sum_{n=1}^{N}\sum_{m=1}^{M} b_{mn} J_n(z_{nm}\rho)e^{jn\phi}$$

In (4), N and M must be sufficiently large to guarantee the convergence. After replacing the field components in the governing equations, the equations are multiplied by testing functions

$$\Psi_{pq}^{*} = J_q(z_{qp}\rho)e^{-jq\phi} \tag{5}$$

and the result is integrated over the whole region. The terms containing the derivatives of permittivity can be integrated using the integration by parts to suppress derivatives of Dirac delta function if the permittivity has discontinuities. Two coupled equation systems will be obtained that if combined to-

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gether, an eigenvalue problem will be made in which β^2's are eigenvalues and the modes are designated by the eigenvectors. In the formulations, symmetries can ease the solution and enhance the accuracy with the same number of harmonics.

III. Numerical Results

As an example the method was used for finding the modes of the step-index rectangular dielectric waveguide which has no analytic solution [7].

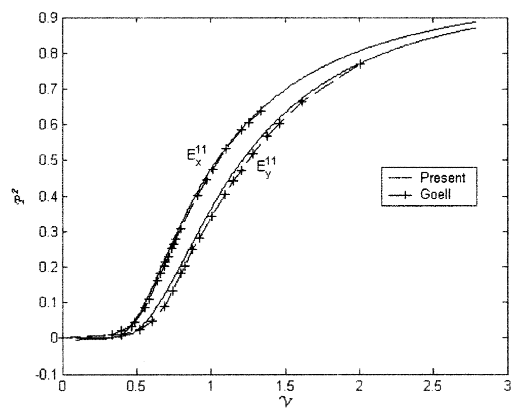
The problem of square dielectric waveguide was solved for the modes for which x=0 is PEC and y=0 is PMC. The results are compared with the results mentioned in [5]. The propagation curves of the E_{11}^x and E_{11}^y modes are plotted for a waveguide having the step refraction index from 1 to 1.5 and an aspect ratio (L_x/L_y) of 2. The mode nomenclature is due to the number of exterma (in x and y directions) of the longitudinal component of the Poynting vector belonging to the mode that has larger x or y component (depending on the superscript.) The plots are drawn in terms of normalized parameters introduced by Goell [4]; normalized frequency:

$$\mathcal{V} = f\frac{2L_y}{c}\sqrt{\varepsilon_r - 1} \qquad (6)$$

, and normalized propagation constant $\mathcal{V}$ $\mathcal{P}$

$$\mathcal{P}^2 = \frac{\varepsilon_{eff} - 1}{\varepsilon_r - 1} \qquad (7)$$

in which f is the frequency, c is the light speed, $\varepsilon_{eff} = \frac{\beta^2}{k_0^2}$.



IV. Conclusion

The methods of Fourier decomposition and mapping to finite regions help to solve the problems of infinite region of definition. As was shown vectorial investigations are possible and good results can be obtained. The drawback of the method is the need for integration time and burden. This could be ignored if the resulted matrices are used for wide range of frequencies and refraction index differences. The ease of mode discovery in addition with the mode distribution makes the method interesting.

References

[1] Marcatili E.A.C, "Dielectric Rectangular Waveguide and Directional Coupler for Integrated Optics," *Bell System Technical Journal*, 1969, pp. 2071-2103

[2] A. Sharma, P.K. Mishra, AND A.K. Ghatak, "Single-Mode Optical Waveguides and Directional Couplers with Rectangular Cross Section: A Simple and Accurate Method of Analysis", Journal Of Lightwave Technology, Vol. 6, No. 6, 1988, pp.1119-1125

[3] Silvester P., "A general high-order finite-element waveguide analysis program", *IEEE Transactions on Microwave Theory and Techniques*, vol. MTT-17, no. 4, April 1969, pp. 204-210

[4] D. Marcuse, "Solution of the Vector Wave Equation for General Dielectric Waveguides by the Galerkin's Method", *IEEE Journal of Quantom Electronics*, Vol. 28, No. 2, 1992.

[5] Goell J.E, "A circular-harmonic computer analysis of rectangular dielectric waveguides", *Bell System Technical Journal*, vol. 48, no. 7, September 1969, pp. 2133-2160.

[6] S. J. Hewlett and F. Ladouceur, "Fourier Decomposition Method Applied to Mapped Infinite Domains: Scalar Analysis of Dielectric Waveguides Down to Modal Cut-off," *Journal of Lightware Technology*, Vol. 13, No.3, 1995.

[7] S. Dudorov, "Rectangular Dielectric Waveguide and Its Optimal Transition to A Metal Waveguide", *PHD Thesis*, Helsinki University, 2002.