A Novel Approach for Direct Kinematics Solution of 3-RRR Parallel Manipulator Following a Trajectory

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Abstract
One of the essential parts in control and simulation of parallel manipulators is obtaining direct kinematic solution. The Direct kinematics of parallel manipulators has been a challenging problem because of complexity and is considered by a few researchers. There is not in general a close-form solution for the problem. Identifying the proper solution among multiple solutions is another challenging problem. This paper presents direct kinematic solutions for a planar 3-RRR parallel manipulator. Numerical methods are traditionally used to obtain one of the solutions which due to path requirement and complexity of the path may not lead to the desired solution. We introduce the use of manipulator's Jacobian in order to estimate the next direct kinematic solution. Jacobian is calculated using current point coordinates in a path to estimate next point coordinates. But it is shown that errors may add up and become non negligible. Bezout's elimination is used to obtain all possible solutions however identification of the correct solution still remains. The proposed method combines Bezout's elimination with manipulator Jacobian to efficiently identify the desired solution.
Keywords: Direct kinematics, Parallel manipulator, Jacobian, Bezout's elimination method.

Introduction
Parallel manipulators are defined as ''a closed-loop kinematic chain mechanisms which end effector is linked to the base by several independent kinematic chains '' [1]. Due to their high stiffness, high speed and large load carrying capacity, parallel mechanisms have become very popular in the past decade. Parallel mechanisms typically consist of two platforms which are connected by several serial kinematic chains. The early design of the parallel manipulator was a six-linear jack system devised as a tire-testing machine proposed by Gough and Whitehall [2]. Stewart [3] designed a general six-legged platform manipulator as an airplane simulator. Since then, parallel mechanisms have been studied extensively by numerous researchers (Hunt [5], Fichter [6], Sugimoto [7], Merlet [1], Nanua et al. [8], Innocenti and Parenti-Castelli [9], Zhang and Song [10], Tsai [11]). Many researchers considered kinematics of parallel manipulators. Kinematics problem have been investigated in two different branches: direct kinematics and inverse kinematics. Mostly there is a close form solution for inverse kinematics but not for direct kinematics. Direct kinematics of parallel manipulator has been studied by a few researchers, [1], [12], [13]. In many industrial applications, such as some assembly and machining operations, parallel manipulators with fewer degrees of freedom than six are successfully used. In this paper, a 3-RRR planar three-degree-of-freedom parallel manipulator is studied. A new method for finding the desired solution for a 3-RRR parallel manipulator has been proposed. It is shown that there exists a maximum of 6 real solutions for the direct kinematics problem [12]. However, obtaining the one desired solution has been a challenging problem. The method combines Bezout's elimination method with the advantages of manipulator's Jacobian to find the desired solution in an efficient and novel manner.

Kinematics of parallel manipulators
The basic components of a robot system include the manipulator, sensory devices, controller, and power conversion unit where the sensory devices are distributed all over the system. Equivalently, a manipulator may be considered to be composed of three parts—the major linkages, the minor linkages (wrist components), and the end effector (gripper or tool).
Kinematic analysis of parallel manipulators includes solution to the direct and inverse kinematic problems as well as velocity and acceleration inversion. The direct kinematics of a robot deal with the computation of the position and orientation of the robot end effector in terms of the robot active joint variables. The joint variables are angles between two links or joint extensions depending on whether the joint is revolute or prismatic. Direct kinematic analysis is one of essential parts in control and simulation of parallel manipulators.
In contrast to the direct kinematics, the inverse kinematics problem of a robot deal with the determination of the joint variables corresponding to any specified position and orientation of the end effector. The inverse kinematics problem is important since manipulation tasks are naturally formulated in terms of the expected end effector position and orientation. In most parallel manipulators solving the direct kinematics problem according to the manipulator's kinematical sub chains and due to the fact

that a systematic general closed-form solution is not readily available is more complex than the inverse kinematics problem.

Structure of planar 3-RRR parallel manipulator

A 3-RRR planar three-degree-of-freedom is studied in this paper. The manipulator consists of three legs. Each leg is made of one actuated revolute joint and two passive revolute joints. It includes three closed kinematics loops where two of them are independent and the other is dependent. The kinematic chain of the general mechanism is shown in "Figure 1".

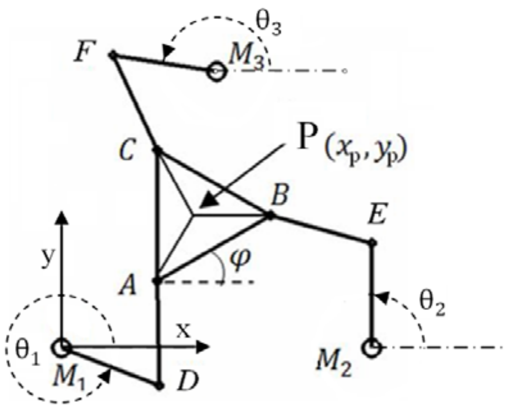

Figure 1: Planar parallel 3-RRR manipulator [2]

The three motors M_1, M_2 and M_3 are fixed and placed on the vertices of an equilateral triangle. This manipulator consists of a kinematic chain with three closed loops, namely M_1DABEM_2 , M_2EBCFM_3 and M_3FCADM_1. The gripper is rigidly attached to the moving base, triangle ABC. It is pointed out that only two of the aforementioned loops are kinematically independent [12]. The gripper will be asked to follow an arbitrary trajectory in the plane of motion. Therefore, there should not be any preferred general orientation for which the manipulator would have better properties. This suggests that the manipulator should be symmetric [12].

To calculate the degrees of freedom of the system we must find the number of the one-DOF joints and the number of movable rigid bodies [14]. The 3-RRR manipulator's structure has nine one-degree-of-freedom joints (m) and seven movable rigid bodies (r). So we can write:

$$n = 6\times r - 5\times m$$
$$= 6\times 7 - 5\times 9 = 3 \quad (1)$$
$$l = m - r$$
$$= 9 - 7 = 2 \quad (2)$$

This results in three degrees-of-freedom (n) and two independent kinematic loops (l).

Direct kinematic problem

The gripper of the 3-RRR parallel manipulator will be asked to follow a desired path. The center position and orientation of the moving platform (gripper) is defined by x_p, y_p and φ, respectively. The direct kinematics problem seeks to obtain the position and orientation of the moving platform given the position of the actuated joint angles. It is shown that there exists a maximum of 6 real solutions for the problem [12]. The closed form solution to the direct kinematics is not accessible. Therefore, the solution for the 3-RRR manipulator requires utilization of a numerical method which depending on the initial guess will lead to one of the six possible solutions. Referring to"Figure1", if the three input angles are specified, the position of points *D, E* and *F* are readily computed. Moreover, the chain *DABE* is a four-bar linkage "Figure 2" of which *C* is a point on the coupler link generating a coupler curve.

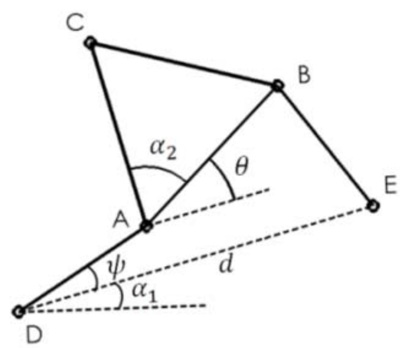

Figure 2: Equivalent four bar mechanism [12]

A solution for the closure of the whole kinematic chain is obtained whenever the coupler curve described by the motion of point *C* intersects the circle defined by the rotation of link *FC* around point *F*. The forgoing principle is now used to derive the equations that will lead to the two following coupled trigonometric equations:

$$x_C = x_D + l_2 \cos(\alpha_1 + \psi) + \sqrt{3} l_3 \cos(\alpha_1 + \alpha_2 + \theta) \quad (3)$$

$$y_C = y_D + l_2 \sin(\alpha_1 + \psi) + \sqrt{3} l_3 \sin(\alpha_1 + \alpha_2 + \theta) \quad (4)$$

Where:

$$\alpha_2 = \pi/3 \quad (5)$$

$$\alpha_1 = \operatorname{atan2}\left[\frac{y_E - y_D}{x_E - x_D}\right] \quad (6)$$

$$\theta_{1,2} = 2\tan^{-1}\left[\frac{B \pm \sqrt{B^2 - AC}}{A}\right] \quad (7)$$

With A, B and C that summarized in the following form:

$$A = m_1 - m_2 + (1 + m_3)\cos(\psi), \quad (8)$$

$$B = \sin(\psi), \quad (9)$$

$$C = m_1 + m_2 + (m_3 - 1)\cos(\psi), \quad (10)$$

Where:

$$m_1 = \frac{-d^2-3l_3^2}{2\sqrt{3}l_2 l_3}, \quad (11)$$

$$m_2 = \frac{d}{l_2}, \quad (12)$$

$$m_3 = \frac{d}{\sqrt{3}l_3}, \quad (13)$$

$$d=\sqrt{(x_E - x_D)^2 + (y_E - y_D)^2} \quad (14)$$

The coupler curve intersects the circle defined by the rotation of link F C around point F. Therefore, following equation could be obtained:

$$(x_C - x_F)^2 + (y_C - y_F)^2 = l_2 \quad (15)$$

Now, the nonlinear relations that must be solved are equations (7) and (15). By rewriting these two equations in terms of θ and ψ variables, a set of non-linear equations will be obtained. These two equations could be solved numerically for angles θ and ψ. Derivative methods such as Secant method are usually suggested to solve these problems [15]. These methods provide only one of the solutions which itself depends on the initial guess. However, considering the requirement of the path following, only one solution is possible which may not be the one provided by the derivative methods. Another approach is using Bezout's method to find all solutions for the direct kinematic problem. However, due to path following only one of the real solutions fits the path and must be selected. We will introduce the use of manipulator's Jacobian in order to estimate the solution while following a path. However, the error may not be negligible for all points along the path. The proposed method combines manipulator's Jacobian along with Bezout's method to identify the one desired solution in an efficient way.

The use of Manipulator Jacobian

The Jacobian relation for the 3-RRR manipulator is defined as following:

$$Jt + K\dot{\theta} = 0 \quad (16)$$

Where t is Cartesian vector, $t = \left[\dot{x_p}, \dot{y_p}, \dot{\varphi}\right]^T$ and θ is the vector of actuated joint angles, $\theta = [\theta_1, \theta_2, \theta_3]^T$. J and K are 3 by 3 matrices which are illustrated below:

$$J = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad (17)$$

$$K = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \quad (18)$$

Where a_i, b_i, c_i and d_i are in terms of platform coordinates and presented by Gosselin and Angeles [12]. Equation (16) may be used to estimate the next position of the moving platform in the form of the following equation:

$$\begin{Bmatrix} x_p \\ y_p \\ \varphi \end{Bmatrix}_{new} = \begin{Bmatrix} x_p \\ y_p \\ \varphi \end{Bmatrix}_{old} - J_{old}^{-1} K_{old}(\theta_{new} - \theta_{old}) \quad (19)$$

Where J_{old} and K_{old} are in the terms of the last state of the manipulator. Next, we will show Jacobian could be used for direct kinematic estimation in a path following problem. It should be noted that workspace of manipulator is a 3 dimensional space which includes x_p, y_p and φ. A certain circular path is selected for x_p and y_p variables. The angle of the moving platform, φ, is assumed to be zero. The following steps are implemented to identify direct kinematics solution:

1. The path is divided into an arbitrary number of points. For each point the coordinate (x_p, y_p and φ) are considered for the next step.
2. Using inverse kinematics [12], active joint variables (θ_1, θ_2 and θ_3) are obtained for entire points along the path.
3. Starting from first point, this point is considered as the old point coordinates, $\begin{Bmatrix} x_p \\ y_p \\ \varphi \end{Bmatrix}_{old}$.
4. J_{old} and K_{old} are calculated and an estimation for next (new) point coordinates, $\begin{Bmatrix} x_p \\ y_p \\ \varphi \end{Bmatrix}_{new}$, is obtained by (19).
5. The new coordinate calculated in step 4 is used as "old" coordinate for obtaining the next solution.
6. Repeating steps 4 through 5 will obtain the direct kinematic solution of entire path.

Results are compared with the desired circular path and depicted in the "Figure 3".

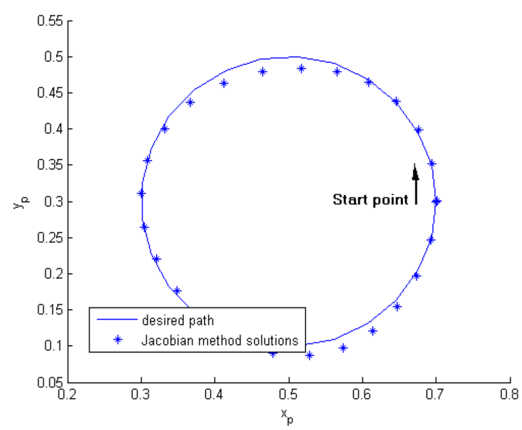

Figure 3: The desired path traced using the manipulator's Jacobian with errors

As is shown in "Figure 3", utilizing Jacobian will produce results that approximately follow the desired path with some points having non negligible errors. However, the Jacobian will aid to identify the correct solution from solutions obtained by Bezout's elimination method, in an efficient manner.

The Bezout's elimination method

The Bezout's elimination method is traditionally used for reducing a set of polynomials of multiple variables into a polynomial of only one variable [16]. To solve the nonlinear relations (7) and (15) by Bezout's elimination method, the trigonometric equations must be transformed into a set of polynomials. This transformation can be achieved by using the following trigonometric identities for change of variables:

$$\tan\left(\frac{\theta}{2}\right) = z_1 \quad (20)$$

$$\tan\left(\frac{\psi}{2}\right) = z_2 \quad (21)$$

$$\sin(\theta) = \frac{2z_1}{1+z_1^2} \quad (22)$$

$$\sin(\psi) = \frac{2z_2}{1+z_2^2} \quad (23)$$

$$\cos(\theta) = \frac{1-z_1^2}{1+z_1^2} \quad (24)$$

$$\cos(\psi) = \frac{1-z_2^2}{1+z_2^2} \quad (25)$$

Substituting the above expressions into relations (7) and (15) and applying some simplifications, one can obtain the following polynomials:

$$f_1 = \sum_{i=1}^{3}\left(\sum_{j=1}^{3}\left(a_{ij} z_1^{i-1} z_2^{j-1}\right)\right) \quad (26)$$

$$f_2 = \sum_{i=1}^{3}\left(\sum_{j=1}^{3}\left(b_{ij} z_1^{i-1} z_2^{j-1}\right)\right) \quad (27)$$

Where a_{ij} and b_{ij} are the coefficient of polynomial equations (26) and (27) which are collected by the powers of z_1 and z_2.

With the Bezout's method, variable z_1 could be eliminated in the equations (7) and (15) and the resulting equation is given as follows:

$$\begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} = 0 \quad (28)$$

Where F_{ij} is defined in following equations:

$$F_{11} = \begin{vmatrix} a_{33}z_2^2 + a_{32}z_2 + a_{31} & a_{13}z_2^2 + a_{12}z_2 + a_{11} \\ b_{33}z_2^2 + b_{32}z_2 + b_{31} & b_{13}z_2^2 + b_{12}z_2 + b_{11} \end{vmatrix} \quad (29)$$

$$F_{12} = \begin{vmatrix} b_{23}z_2^2 + b_{22}z_2 + b_{21} & a_{23}z_2^2 + a_{22}z_2 + a_{21} \\ b_{33}z_2^2 + b_{32}z_2 + b_{31} & a_{33}z_2^2 + a_{32}z_2 + a_{31} \end{vmatrix} \quad (30)$$

$$F_{21} = \begin{vmatrix} a_{23}z_2^2 + a_{22}z_2 + a_{21} & a_{13}z_2^2 + a_{12}z_2 + a_{11} \\ b_{23}z_2^2 + b_{22}z_2 + b_{21} & b_{13}z_2^2 + b_{12}z_2 + b_{11} \end{vmatrix} \quad (31)$$

$$F_{22} = \begin{vmatrix} a_{33}z_2^2 + a_{32}z_2 + a_{31} & a_{13}z_2^2 + a_{12}z_2 + a_{11} \\ b_{33}z_2^2 + b_{32}z_2 + b_{31} & b_{13}z_2^2 + b_{12}z_2 + b_{11} \end{vmatrix} \quad (32)$$

Where $|*|$ denotes the determinant of a matrix. After expanding and simplification, relation (28) becomes:

$$L_9 z_2^8 + L_8 z_2^7 + L_7 z_2^6 + L_6 z_2^5 + L_5 z_2^4 + L_4 z_2^3 + L_3 z_2^2 + L_2 z_2 + L_1 = 0 \quad (33)$$

Where L_i are the functions of a_{ij} and b_{ij} only. Detailed expressions of L_i could be developed by expanding equation (33).

The eight solutions for z_2 could be found numerically by solving equation (33). Substituting each solution of z_2 in converted form of equations (7) and (15), that are equations (26) and (27), results in two solutions for z_1. Therefore, a total of 16 solutions will be obtained. As it was stated earlier, a maximum of 6 solutions for direct kinematics exists. Therefore there exist maximum of six real solutions and the rest are imaginary. θ and ψ can now be calculated.

$$\theta = 2\operatorname{atan}(z_1), \quad (34)$$

$$\psi = 2\operatorname{atan}(z_2) \quad (35)$$

Then platform position and orientation could be obtained as following:

$$x = x_D + l_2\cos(\alpha_1 + \psi) + l_3\cos(\alpha_1 + \alpha_2/2 + \theta) \quad (36)$$

$$y = y_D + l_2\sin(\alpha_1 + \psi) + l_3\sin(\alpha_1 + \alpha_2/2 + \theta) \quad (37)$$

$$\varphi = \psi + \theta \quad (38)$$

An example is performed for the manipulator with properties shown in "Table 1".

Table 1: Manipulator properties

	First leg	Second leg	Third leg
l_1 (m)	0.5	0.5	0.2
l_2 (m)	0.5	0.5	0.2
l_3 (m)	0.5	0.5	0.2
Actuator's position (m)	$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$	$\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$	$\begin{Bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{Bmatrix}$
Actuator's angle (degrees)	-89.825	-6.778	36.986

Direct kinematics results are shown in "Table 2".

Where in each symmetric leg, lengths for the first and second links are l_1 and l_2. The platform equilateral triangle is considered as the third link with the length l_3 that is the distance between each vertex to the center of surface of the third link.

Table 2: The positions obtained by using Bezout's elimination method

Solution Number	x_p (m)	y_p (m)	φ_p (radian)
1	-	-	-
2	-	-	-
3	-	-	-
4	-	-	-
5	-	-	-
6	-	-	-
7	0.8000	0.7000	0.5236
8	-	-	-
9	-	-	-
10	-	-	-
11	-	-	-
12	-	-	-
13	1.0739	-0.0665	0.1399
14	-	-	-
15	-	-	-
16	0.9278	0.3581	-1.2159

A similar path as outlined in the previous section is considered to illustrate the results of Bezout's method. As before the path is divided into arbitrary number of points. For each point, inverse kinematics is implemented to obtain corresponding actuated joint values (θ_1, θ_2 and θ_3). These values are used as inputs to the Bezout's method to obtain all solutions for the center position and orientation of the platform, x_p , y_p and φ. Bezout's results for entire path are depicted in "Figure 4". As it shown, in each iteration one of the answers suits the desired path and the path will be covered by at least one correct answer per iteration.

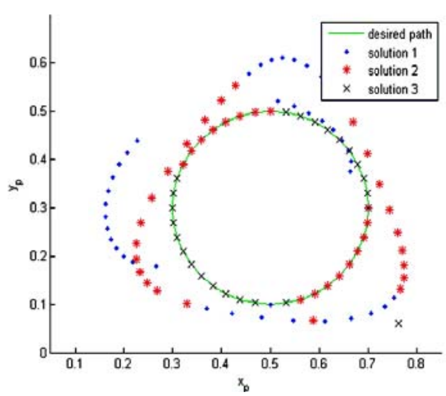

Figure 4: Bezout's method solutions on a circular path

Therefore there must be a method to collect the appropriate answer that falls on the desired path.

Manipulator's Jacobian and Bezout's methods

As stated before, only one solution among solutions obtained from the Bezout's method fits the desired path and must be selected. To find this unique solution, the Jacobian method could be utilized. To do this the same circular path as the previous sections is chosen. The following procedure is use for finding direct kinematics solutions for entire path:

1. Divide the path into an arbitrary number of points.
2. Use inverse kinematics to find motor angles for all points along the path.
3. Start from the first point on the path.
4. Use both motor angles and platform position of the current point and utilize Jacobian to estimate the next point coordinates.
5. Obtain all direct kinematic results for next point using Bezout's elimination method.
6. Calculated the distance between each solution found by the Bezout's method and the solution found by the Jacobian method.
7. The solution that offers the minimum distance is the desired solution.
8. Go back to step 4 for next point

The proposed approach is depicted in "Figure 5".

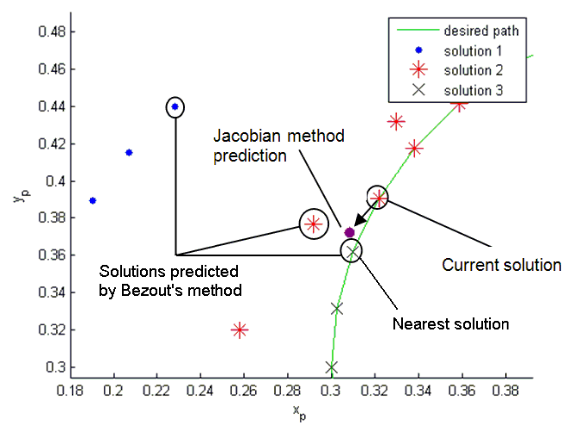

Figure 5: Identification of the nearest result

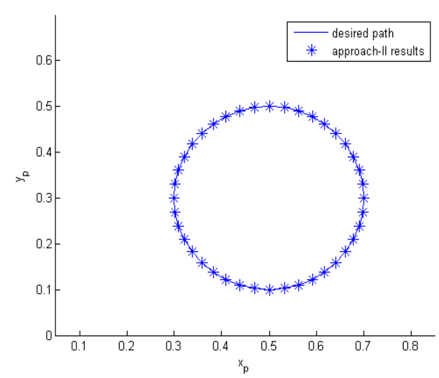

Figure 6: Manipulator's path coverage verified

The proposed algorithm is verified on the same circular trajectory. Results are depicted in “Figure 6”. This figure shows that the solutions of the combine method follow the trajectory.

Conclusion

In this paper a planar 3-RRR parallel manipulator undergoing a desired path was introduced. First the direct kinematics equations of the manipulator were derived. A method that utilized manipulator's Jacobian was introduced to approximate the next solution on the path. A circular path was selected to verify this method. Results indicated that errors are non negligible in some circumstances. Next Bezout's elimination was used to obtain all possible solutions for all platform coordinates along the path. Finally, Bezout's elimination and manipulator's Jacobian were combined. This allows identification of the one desired solution that fits the path. The proposed method was implemented on same circular path. Results confirm correctness of the combined method.

List of symbols

J	The first Jacobian matrix of the parallel manipulator
K	The first Jacobian matrix of the parallel manipulator
l	Number of independent kinematic loops in parallel mechanism
l_1	Length of the first links in each leg
l_2	Length of the second links in each leg
l_3	The distance between moving platform’s apex to it’s centre of surface
m	Number of movable rigid bodies in the mechanism
n	The degrees of freedom of parallel mechanism
r	Number of one DOF joints in the mechanism
x_p	The x coordinate of moving platform position
y_p	The y coordinate of moving platform position
x_{2i}	The x coordinate of the third joint on each leg
y_{2i}	The y coordinate of the third joint on each leg
φ	Moving platform pure rotation about z direction
θ	Rotation of moving platform according to line DE
α_i	Rotation of line DE
ψ_i	Rotation of link DA according to line DE

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