Theorem 2.4. *Let X and K be compact plane sets, $K \subset X$ and $m(X\backslash K) = 0$, where m is the planar measure. Then $lip_{RX}(X, K, \alpha) = lip_R(X, K, \alpha)$.*

As an application of the above theorem we have the following interesting results, which have already been proved by the authors, adopting a different method [preprint].

Corollary 2.5. *If $m(X\backslash K) = 0$ then $R(X, K) = R(X)$. In particular, if $m(X) = 0$ then $R(X) = C(X)$, which is, in fact, the Hartogs-Rosenthal Theorem.*

Corollary 2.6. *Let X be a compact plane set and S and T be compact subsets of X such that $m(S + T) = 0$, where $S + T$ is the symmetric difference of S and T. Then $R(X, S) = R(X, T)$.*

Remark 2.7. One might think that if $m(X\backslash K) = 0$ then $lip_{PX}(X, K, \alpha) = lip_P(X, K, \alpha)$. But this is not true, as the following example shows: Take $X = \mathbb{T} = \{z \in C : |z| = 1\}$ and $K = \{z \in X : Imz \geq 0\}$. Then $lip_{PX}(X, K, \alpha) \neq lip_P(X, K, \alpha)$.

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THE TOPOLOGICAL CENTRE OF ELLIS GROUPS, AND WEYL ALGEBRAS

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Abstract. After introducing a new class of distal dynamical systems and investigating the topological properties of these systems we characterize the topological centre of the Ellis groups corresponding to the newly defined transformations. The dynamical systems under review are natural extensions of the dynamical systems corresponding to the elements of the Weyl algebra on integers introduced by E. Salehi in [4]. Because of the importance we feel for the Weyl algebra on integers, we give a generalization of this algebra to arbitrary semitopological semigroups, and at the same time we show that the elements of the involved algebra are distal.

1. Introduction and Preliminaries

For the background and notations we follow Berglund et al. [1], however for the benefit of the reader we mention some necessary materials. A dynamical system is a pair (X, T) where X is a compact Hausdorff space and T is a homeomorphism from X onto X. The enveloping semigroup $\Sigma(X, T)$ of a dynamical system (X, T) was defined by R. Ellis as the closure of the set $\{T^m : m \in \mathbb{Z}\}$ in X^X with the relativization of the product topology from X^X. Σ is a compact right topological semigroup under the relative product topology from X^X and composition multiplication. It is an interesting result of Ellis that the enveloping semigroup of a

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dynamical system (X,T) is a compact right topological group if and only if (X,T) is distal, that is $\lim_\alpha T^{m_\alpha}x = \lim_\alpha T^{m_\alpha}y$, for some net $\{m_\alpha\}$ in $\mathbb{Z}$ and for some x,y in X, implies $x = y$. The enveloping semigroup of a distal dynamical system is called the Ellis group of the dynamical system.

An interesting problem concerning any compact right topological semigroup S is the problem of characterizing its topological centre $\Lambda(S) = \{s \in S; l_s : S \to S : t \mapsto st \text{ is continuous}\}$. For example, the enveloping semigroup Σ of any dynamical system (X,T) gives rise to a compact right topological semigroup. Obviously, in such a case, $\Lambda(\Sigma)$ contains the set $\{T^m : m \in \mathbb{Z}\}$.

Let $\mathbb{Z}, \mathbb{T}$ and $\mathbb{E}$ be the group of integers, the circle group and the set of all endomorphisms of $\mathbb{T}$, respectively. In his extensive work [3], Namioka studied the distal flow $(\mathbb{Z}, \mathbb{T}^2)$ (under a certain group action) and by giving a concrete representation, he identified its Ellis group with $\mathbb{E} \times \mathbb{T}$ and also showed that its topological centre is equal to the set $\{((\ )^n, z) : n \in \mathbb{Z}, z \in \mathbb{T}\}$. Also Milnes [2] continued the methods of [3] and studied the Ellis groups of a wide variety of distal flows.

Following [2] and [3], we study the Ellis group Σ of a specific distal flow $(\mathbb{Z}, \mathbb{T}^k)$ by embedding it into $\mathbb{E}^{k-1} \times \mathbb{T}$, and we show that $\Lambda(\Sigma)$ (for $k \neq 1$) is equal to the set $\{((\ )^n, (\ )^{n^2}, ..., (\ )^{n^{k-1}}, z) : n \in \mathbb{Z} \text{ and } z \in \mathbb{T}\}$ in $\mathbb{E}^{k-1} \times \mathbb{T}$.

Milnes, in example 3 of [2], used the same flow, for the special case $k = 4$, to give a description of a flow arising from the distal function $f(n) = \lambda^{n^4}$ on $\mathbb{Z}$, where $\lambda \in \mathbb{T}$ is not a root of unity.

For a semitopological semigroup S, the space of all bounded continuous complex valued functions on S is denoted by $C(S)$. For $f \in C(S)$ and $s \in S$ the right (respectively, left) translation of f by s is the function $R_s f = f \circ r_s$ (respectively, $L_s f = f \circ l_s$). A left translation invariant unital C^*-subalgebra F of $C(S)$ (i.e., $L_s f \in F$ for all $s \in S$ and $f \in F$) is called m-admissible if the function $s \mapsto (T_\mu f)(s) = \mu(L_s f)$ belongs to F for all $f \in F$ and $\mu \in S^F$(=the spectrum of F). If F is m-admissible then S^F under the multiplication $\mu\nu = \mu \circ T_\nu$ ($\mu, \nu \in S^F$), furnished with the Gelfand topology is a compact Hausdorff right topological semigroup and it makes S^F a compactification (called the F-compactification) of S.
Some of the usual m-admissible subalgebras of $C(S)$, that are needed in the sequel, are the left multiplicatively continuous, and the distal functions on S. These are

denoted by $LMC(S)$ and $D(S)$, respectively. The norm closure of the algeb generated by the set $\{n \mapsto \lambda^{n^k} : \lambda \in \mathbb{T} \text{ and } k \in \mathbb{N}\}$ of functions on $(\mathbb{Z}, +)$ wa called the Weyl algebra by E. Salehi in [4]. By giving a generalization of th weyl algebra to arbitrary semitopological semigroups, we show that the involve algebras are m-admissible and distal.

2. Main results

2.1. The topological centre of Ellis groups.

Definition 2.1. Let λ be a fixed irrational element of $\mathbb{T}$ (i.e. λ is not a root o unity) and fix $k \in \mathbb{N}$. Define $T : \mathbb{T}^k \to \mathbb{T}^k$ by

$$T(\omega_1, \omega_2, \dots, \omega_k) = (\nu_1, \nu_2, \dots, \nu_k)$$

where for each $i = 1, 2, \dots, k$

$$\nu_i = \lambda^{\binom{k}{i}} \prod_{j=1}^{i} \omega_j^{\binom{k-j}{i-j}}.$$

A sequence $\{x_n\}$ in a compact abelian group G with the normalized Haa measure μ is said to be uniformly distributed in G if for each continuous comple valued function f on G, $lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_G f d\mu$. It is clear that ever uniformly distributed sequence in G is dense in G. Let $e = (1, 1, \dots, 1)$, then fo each $m \in \mathbb{Z}$, $T^m(e) = (\lambda^{\binom{k}{1}m}, \lambda^{\binom{k}{2}m^2}, \dots, \lambda^{m^k})$.

Lemma 2.2. *The points $T^m(e)$ $(m \geq 1)$ are uniformly distributed, and so dens in $\mathbb{T}^k$.*

This lemma is proved easily by using a famous result of H. Weyl [5].

The next proposition, which verifies the distality and minimality of the latte flow, is an easy verification:

Proposition 2.3. *The flow $(\mathbb{Z}, \mathbb{T}^k)$ is a distal minimal flow.*

If $\sigma \in \Sigma(\mathbb{Z}, \mathbb{T}^k)$ and $\sigma = \lim_\alpha m_\alpha$, for some net $\{m_\alpha\}$ in $\mathbb{Z}$, by taking a subne of $\{m_\alpha\}$ if necessary, we may assume that $\lim_\alpha \lambda^{m_\alpha^k} = z$ and $\lim_\alpha \eta^{m_\alpha^i} = \theta_i(\eta)$, fo $i = 1, 2, ..., k-1$ and every $\eta \in \mathbb{T}$. Therefore $\sigma((\omega_1, \omega_2, \dots, \omega_k)) = (z_1, z_2, ..., z_k)$ where $z_i = \theta_i(\lambda^{\binom{k}{i}}) \prod_{j=1}^{i} \theta_{i-j}(\omega_j^{\binom{k-j}{i-j}})$, for $i = 1, 2, ..., k-1$, (with $\theta_0 = id_{\mathbb{T}}$ i mind), and $z_k = z \prod_{j=1}^{k} \theta_{k-j}(\omega_j)$. These observations lead us to the fact that each $\sigma \in \Sigma$ corresponds to a k-fold $(\theta_1, ..., \theta_{k-1}, z) \in \mathbb{E}^{k-1} \times \mathbb{T}$. In fact we hav the following lemma:

Lemma 2.4. *If $\Theta : \Sigma(\mathbb{Z}, \mathbb{T}^k) \to \mathbb{E}^{k-1} \times \mathbb{T}$ is defined by $\Theta(\sigma) = (\theta_1, ..., \theta_{k-1}, z)$, where $\sigma, \theta_1, ..., \theta_{k-1}$ and z are as given above, then Θ is an embedding isomorphism into $\mathbb{E}^{k-1} \times \mathbb{T}$.*

Lemma 2.5. *Let the mapping $\Theta : \Sigma(\mathbb{Z}, \mathbb{T}^k) \to \mathbb{E}^{k-1} \times \mathbb{T}$ be defined as in Lemma 2.4. Let $\sigma \in \Lambda(\Sigma(\mathbb{Z}, \mathbb{T}^k))$ and let $\Theta(\sigma) = (\theta_1, ..., \theta_{k-1}, z)$. Then for each $i = 1, 2, ..., k-1$, θ_i is continuous.*

We are now sufficiently prepared for the main theorem on the topological centre of $\Sigma = \Sigma(\mathbb{Z}, \mathbb{T}^k)$.

Theorem 2.6. *For $1 \neq k \in \mathbb{N}$, $\Lambda(\Sigma(\mathbb{Z}, \mathbb{T}^k)) = \{((\ )^n, (\ )^{n^2}, ..., (\ )^{n^{k-1}}, z) : n \in \mathbb{Z}$ and $z \in \mathbb{T}\}$.*

Proof. Let $\sigma = \lim_\alpha m_\alpha$ be in $\Lambda(\Sigma(\mathbb{Z}, \mathbb{T}^k))$ and let the mapping $\Theta : \Sigma(\mathbb{Z}, \mathbb{T}^k) \to \mathbb{E}^{k-1} \times \mathbb{T}$ be defined as in Lemma 2.4 and let $\theta(\sigma) = (\theta_1, ..., \theta_{k-1}, z)$. The continuity of θ_j's (Lemma 2.5) implies that $\theta_j = (\ )^{n_j}$, for some $n_1, n_2, ..., n_{k-1} \in \mathbb{Z}$. We are going to show that $(\ )^{n_j} = (\ )^{n_1{}^j}$, for every $j = 2, ..., k-1$. Fix p as an arbitrary prime, then for $\eta = e^{\frac{2\pi i}{p}}$ since $(\eta)^{m_\alpha} \to (\eta)^{n_1}$, we get $m_\alpha = n_1 \pmod p$, eventually. Hence for each $j = 2, ..., k-1$, $(m_\alpha)^j = n_1{}^j \pmod p$, eventually. Since p was an arbitrary prime, $(m_\alpha)^j = n_1{}^j$, eventually. On the other hand one can show similarly that $(m_\alpha)^j = n_j$, eventually. Hence $n_j = n^j$ for each $j = 1, ..., k-1$, and so $\sigma = ((\ )^{n_1}, (\ )^{n_1^2}, ..., (\ )^{n_1^{k-1}}, z)$ in which $z = \lim_\alpha \lambda^{m_\alpha^k}$. □

2.2. Weyl algebras.

Definition 2.7. Let $\Sigma = \{T_\mu : LMC(S) \to LMC(S); \mu \in S^{LMC}\}$. Let F_0 be the set of all constant functions of modulus 1. For every $k \in \mathbb{N}$ assume that we have defined F_i for $i = 1, 2, \ldots, k-1$ and define F_k by

$F_k = \{f \in LMC : |f| = 1$ and for every $\sigma \in \Sigma, \sigma(f) = f_\sigma f$, for some $f_\sigma \in F_{k-1}\}$;

It is clear from definitions that $F_k \subseteq F_{k+1}$, for all $k \in \mathbb{N} \cup \{0\}$. Let W_k and W be the norm closure of the algebras generated by F_k and $\bigcup_{k \in \mathbb{N}} F_k$ in $C(S)$, respectively.

Lemma 2.8. *The set F_k is left translation invariant and it is also invariant under Σ; in other words, $L_S(F_k) \subseteq F_k$ and $\Sigma(F_k) \subseteq F_k$.*

Lemma 2.9. *All elements of F_k remain fixed under the idempotents of Σ.*

Lemma 2.10. $F_k \subseteq D$.

Theorem 2.11. *For every semitopological semigroup S, W_k and W are m-admis subalgebras of $D(S)$.*

Proof. For the m-admissibility of W_k it is enough to show that it is left translatio invariant and also invariant under Σ. Let $\langle F_k \rangle$ be the algebra generated by F Lemma 2.8 implies that $L_S(\langle F_k \rangle) \subseteq \langle F_k \rangle$ and also $\Sigma(\langle F_k \rangle) \subseteq \langle F_k \rangle$. For ever $f \in W_k$ there exists a sequence $\{f_n\} \subseteq \langle F_k \rangle$ which converges (in the norm of $C(S)$ to f. Let $\sigma \in \Sigma$ and $s \in S$, then the inequalities $\|L_s(f_n) - L_s(f)\| \leq \|f_n - f$ and $\|\sigma(f_n) - \sigma(f)\| \leq \|f_n - f\|$ imply that $L_s(f_n) \to L_s(f)$ and $\sigma(f_n) \to \sigma(f$ respectively. Since for each $n \in \mathbb{N}$, $L_s(f_n)$ and $\sigma(f_n)$ lie in $\langle F_k \rangle$, we have $L_s(f)$ W_k and also $\sigma(f) \in W_k$. It follows that W_k is m-admissible. A similar argumen may apply for the m-admissibility of W. The fact that W_k and W are containe in D follows trivially from Lemma 2.10. □

Also we have the following proposition which generalizes, in part, a result of M Filali.

Proposition 2.12. *If R is a countable discrete ring, then for each character χ o the discrete additive group of R the function $\chi(q(t))$, in which $q(t)$ is a polynomia with coefficients in R, belongs to $W(R_d, +)$ and is also a distal function.*

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