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ARENS REGULARITY OF BOUNDED BILINEAR MAPPINGS AND THEIR THIRD ADJOINTS

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ABSTRACT. We demonstrate a simple criterion for the Arens regularity of a bounded bilinear mapping on normed spaces. Also we provide a requirement for Arens regularity of the second dual of a regular Banach algebra. As a consequence of these results we show that the second dual of the trace-class operators algebra is regular. Also for a compact group G, the algebra $L^\infty(G)$ with convolution multiplication is regular. Some inclusion relations for topological centers of module actions are also included.

1. Introduction and Preliminaries

The adjoint and the Arens regularity of a bilinear map was first raised by Arens in 1951 [1]. In this paper as the main results we introduce criterions for the regularity of a bounded bilinear map and its third adjoint and then state some of their consequences.

Let X, Y and Z be normed spaces and let $f : X \times Y \longrightarrow Z$ be a bounded bilinear map. The adjoints $f^* : Z^* \times X \longrightarrow Y^*$, $f^{**} : Y^{**} \times Z^* \longrightarrow X^*$ and $f^{***} : X^{**} \times Y^{**} \longrightarrow Z^{**}$ of f are defined so that, for $x \in X, y \in Y, z^* \in Z^*, x^{**} \in X^{**}, y^{**} \in Y^{**}$ and $z^{**} \in Z^{**}$,

$$\langle f^*(z^*, x), y\rangle = \langle z^*, f(x, y)\rangle,$$

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$$\langle f^{**}(y^{**}, z^*), x\rangle = \langle y^{**}, f^*(z^*, x)\rangle \quad \text{and}$$
$$\langle f^{***}(x^{**}, y^{**}), z^*\rangle = \langle x^{**}, f^{**}(y^{**}, z^*)\rangle.$$

The higher rank adjoints of f may be defined similarly. All adjoints are trivially bounded bilinear mappings. We also denote by f^r the flip map of f, that is the bounded bilinear map $f^r : Y \times X \longrightarrow Z$ defined by $f^r(y, x) = f(x, y)$ $(x \in X, y \in Y)$. The map f is called Arens regular when the equality $f^{***} = f^{r***r}$ holds.
The first and second topological centers of a bounded bilinear mapping $f : X \times Y \longrightarrow Z$ is defined as follows respectively:

$$Z(f) = \{x^{**} \in X^{**}; y^{**} \longrightarrow f^{***}(x^{**}, y^{**}) : Y^{**} \longrightarrow Z^{**} is\ w^* - w^* - continuous\}$$

and

$$Z^t(f) = \{y^{**} \in Y^{**}; x^{**} \longrightarrow f^{r***r}(x^{**}, y^{**}) : X^{**} \longrightarrow Z^{**} is\ w^* - w^* - continuous\}$$

Let A be a Banach algebra with multiplication π. Then π^{***} and π^{r***r} are actually the first and second Arens products [1], which will be denoted by $\Box$ and $\Diamond$, respectively.

Let A be a Banach algebra, X be a Banach space and let $\pi_1 : A \times X \longrightarrow X$ be bounded bilinear map. Then the pair (π_1, X) is said to be a left Banach $A-$module if $\pi_1(ab, x) = \pi_1(a, \pi_1(b, x))$, for every $a, b \in A, x \in X$. A right Banach $A-$module (X, π_2) may be defined similarly. A triple (π_1, X, π_2) is said to be a Banach $A-$module if (π_1, X) and (X, π_2) are, respectively, left and right Banach $A-$modules and for every $a, b \in A, x \in X$,

$$\pi_1(a, \pi_2(x, b)) = \pi_2(\pi_1(a, x), b).$$

When (π_1, X) is a left Banach $A-$module then a bounded net $\{e_\alpha\}$ in A is said to be a left approximate identity for X, if $\pi_1(e_\alpha, x) \longrightarrow x$, for each $x \in X$. A bounded right approximate identity for X is defined similarly.

The first and the second results of this lecture are taken from the article [5]. Also For terminology and background materials we follow [3], as far as possible.

2. Main Results

The following theorem which is a criterion for the regularity of a bounded bilinear map is taken from [5].

Theorem 2.1. *For a bounded bilinear map $f : X \times Y \longrightarrow Z$ the following statements are equivalent.*

(*i*) *f is regular.*

(*ii*) $f^{****}(Z^*, X^{**}) \subseteq Y^*$.

(*iii*) *The linear map* $x \longmapsto f^*(z^*, x) : X \longrightarrow Y^*$ *is weakly compact for every* $z^* \in Z^*$.

Corollary 2.2. *Let* $f : X \times Y \longrightarrow Z$ *and* $g : X \times W \longrightarrow Z$ *be bounded bilinear mappings and let* $h : Y \longrightarrow W$ *be a bounded linear mapping such that* $f(x, y) = g(x, h(y))$, *for all* $x \in X, y \in Y$. *If* h *is weakly compact, then both* f *and* f^{r*} *are regular.*

The above corollary is an extension of the main result of the paper [2] by Arikan; In fact she deduced merely the regularity of f. As an application of her theorem she has proved that the algebra $L^\infty(G)$ with convolution, where G is a compact group, and the trace-class algebra are regular. Now the next theorem reveals that in the conditions of corollary 2.2, f^{***} will be regular and so by Arikan's method we conclude that the second dual of the recent mentioned algebras are also regular.

Theorem 2.3. *Let* $f : X \times Y \longrightarrow Z$ *be an Arens regular bounded bilinear map. If* f^* *or* f^{r*} *is regular then* f^{***} *is regular.*

Corollary 2.4. *For a compact group* G *the second dual of the algebra* $L^\infty(G)$ *with convolution multiplication is regular.*

Corollary 2.5. *The second dual of the Banach algebra of trace-class operators on a Hilbert space* H *is regular.*

In the sequel we consider A as a Banach algebra. A Banach $A-$module (π_1, X, π_2) is said to be pseudo unital if $\pi_1(A, X) = X$ and $\pi_2(X, A) = X$.The following proposition states some conditions under which the module actions of a Banach $A-$module X are regular.

Proposition 2.6. *Let* A *be Arens regular Banach algebra and* (π_1, X, π_2) *be a pseudo unital Banach* $A-$*module. If* π_2^* *or* π_1^{r*} *is regular then both* π_1 *and* π_2 *are regular.*

We proceed this talk with some relations between topological centers of some bounded bilinear mappings.

Theorem 2.7. *Let* A *be a Banach algebra with product* π *and let* X *and* Y *be normed spaces. If* A *has a bounded left approximate identity (blai) and* $f : A \times X \longrightarrow Y$ *is a bounded bilinear map then* $Z^t(\pi^{r*}) \subseteq Z(f)$.

It is obvious that a similar argument holds when A has a bounded right approximate identity (brai).

Theorem 2.8. *Let* A *be a Banach algebra and let* (π_1, X) *be a left Banach* $A-$*module. Let* Y *be a normed space and* $f : X \times A \longrightarrow Y$ *be a bounded bilinear map. If* A *has a blai for* X *then* $Z^t(\pi_1^{r*}) \subseteq Z(f)$.

Corollary 2.9. *Let* A *be a Banach algebra and let* (π_1, X) *and* (X, π_2) *be a left and a right Banach* $A-$*module. If* A *has a blai for* X *then* $Z^t(\pi_1^{r*}) \subseteq Z(\pi_2) \bigcap Z^t(\pi_1)$

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