Journal of Algebra 320 (2008) 3269–3277

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On nilpotent multipliers of some verbal products of groups

Azam Hokmabadi, Behrooz Mashayekhy *

Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, PO Box 1159-91775, Mashhad, Iran

ARTICLE INFO

Article history:
Received 6 March 2008
Communicated by E.I. Khukhro

Keywords:
Nilpotent multiplier
Verbal product
Nilpotent product
Cyclic group

ABSTRACT

The paper is devoted to finding a homomorphic image for the c-nilpotent multiplier of the verbal product of a family of groups with respect to a variety $\mathcal{V}$ when $\mathcal{V} \subseteq \mathcal{N}_c$ or $\mathcal{N}_c \subseteq \mathcal{V}$. Also a structure of the c-nilpotent multiplier of a special case of the verbal product, the nilpotent product, of cyclic groups is given. In fact, we present an explicit formula for the c-nilpotent multiplier of the nth nilpotent product of the group $G = \mathbf{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbf{Z} \overset{n}{*} \mathbf{Z}_{r_1} \overset{n}{*} \cdots \overset{n}{*} \mathbf{Z}_{r_t}$, where r_{i+1} divides r_i for all i, $1 \leqslant i \leqslant t-1$, and $(p, r_1) = 1$ for any prime p less than or equal to $n+c$, for all positive integers n, c.

1. Introduction and motivation

Let $G = F/R$ be a free presentation of a group G. Then the Baer invariant of G with respect to the variety $\mathcal{N}_c$ of nilpotent groups of class at most $c \geqslant 1$, denoted by $\mathcal{N}_c M(G)$, is defined to be

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, {}_c F]}.$$

$\mathcal{N}_c M(G)$ is also called the c-nilpotent multiplier of G. Clearly if $c = 1$, then $\mathcal{N}_c = \mathcal{A}$ is the variety of all abelian groups and the Baer invariant of G with respect to this variety is

$$M(G) = \frac{R \cap F'}{[R, F]},$$

which is the well-known Schur multiplier of G.

* Corresponding author.
E-mail address: mashaf@math.um.ac.ir (B. Mashayekhy).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.07.013

It is important to find structures for the Schur multiplier and its generalization, the c-nilpotent multiplier, of some famous products of groups. Determining these Baer invariants of a given group is known to be very useful for the classification of groups into isoclinism classes (see [1]).

In 1907, Schur [17], using a representation method, found a structure for the Schur multiplier of a direct product of two groups. Also, Wiegold [19] obtained the same result by some properties of covering groups. In 1979 Moghaddam [13] found a formula for the c-nilpotent multiplier of a direct product of two groups, where $c+1$ is a prime number or 4. Also, in 1998 Ellis [2] extended the formula for all $c \geqslant 1$. In 1997 the second author and Moghaddam [10] presented an explicit formula for the c-nilpotent multiplier of a finite abelian group for any $c \geqslant 1$. It is known that the direct product is a special case of the nilpotent product and we know that regular and verbal products are generalizations of the nilpotent product.

In 1972, Haebich [6] found a formula for the Schur multiplier of a regular product of a family of groups. Then the second author [9] extended the result to find a homomorphic image with a structure similar to Haebich's type for the c-nilpotent multiplier of a nilpotent product of a family of groups.

In section two, we extend the above result and find a homomorphic image for the c-nilpotent multiplier of a verbal product of a family of groups with respect to a variety $\mathcal{V}$ when $\mathcal{V} \subseteq \mathcal{N}_c$ or $\mathcal{N}_c \subseteq \mathcal{V}$.

A special case of the verbal product of groups whose nilpotent multiplier has been studied more than others is the nilpotent product of cyclic groups. In 1992, Gupta and Moghaddam [5] calculated the c-nilpotent multiplier of the nilpotent dihedral group of class n, i.e. $G_n \cong \mathbf{Z}_2 \overset{n}{*} \mathbf{Z}_2$. (Note that in 2001 Ellis [3] remarked that there is a slip in the statement and gave the correct one.) In 2003, Moghaddam, the second author and Kayvanfar [14] extended the previous result and calculated the c-nilpotent multiplier of the nth nilpotent product of cyclic groups for $n = 2, 3, 4$ under some conditions. Also, the second author and Parvizi [11,12] presented structures for some Baer invariants of a free nilpotent group that is the nilpotent product of infinite cyclic groups. Finally the authors and Mohammadzadeh [8] obtained an explicit formula for the c-nilpotent multiplier of the nth nilpotent product of some cyclic groups $G = \mathbf{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbf{Z} \overset{n}{*} \mathbf{Z}_{r_1} \overset{n}{*} \cdots \overset{n}{*} \mathbf{Z}_{r_t}$, where r_{i+1} divides r_i for all i, $1 \leqslant i \leqslant t-1$, for $c \geqslant n$ such that $(p, r_1) = 1$ for any prime p less than or equal to n.

In section three, we give an explicit formula for the c-nilpotent multiplier of the above group G when $(p, r_1) = 1$ for any prime p less than or equal to $n+c$, for all positive integers c, n.

2. Verbal products

A group G is said to be a *regular product* of its subgroups A_i, $i \in I$, where I is an ordered set, if the following two conditions hold:

(i) $G = \langle A_i \mid i \in I \rangle$;
(ii) $A_i \cap \hat{A}_i = 1$ for all $i \in I$, where $\hat{A}_i = \langle A_j \mid j \in I,\ j \neq i \rangle^G$.

Definition 2.1. Consider the map

$$\psi : \prod_{i \in I}^{*} A_i \to \prod_{i \in I}^{\times} A_i,$$

$$a_1 a_2 \ldots a_n \mapsto (a_1, a_2, \ldots, a_n),$$

which is a natural map from the free product of $\{A_i\}_{i \in I}$ on to the direct product of $\{A_i\}_{i \in I}$. Clearly its kernel is the normal closure of

$$\big\langle [A_i, A_j] \bigm| i, j \in I,\ i \neq j \big\rangle$$

in the free product $A = \prod_{i \in I}^{*} A_i$. It is denoted by $[A_i^A]$ and called *the Cartesian subgroup* of the free product (see [16] for the properties of Cartesian subgroups).

The following theorem gives a characterization of a regular product.

Theorem 2.2. *(See Golovin [4].) Suppose that a group G is generated by a family $\{A_i \mid i \in I\}$ of its subgroups, where I is an ordered set. Then G is a regular product of the A_i if and only if every element of G can be written uniquely as a product*

$$a_1 a_2 \dots a_n u,$$

where $1 \neq a_i \in A_{\lambda_i}$, $\lambda_1 < \cdots < \lambda_n$ and $u \in [A_i^G] = \langle [A_i^G, A_j^G] \mid i, j \in I,\ i \neq j \rangle$.

Definition 2.3. Let $\mathcal{V}$ be a variety of groups defined by a set of laws V. Then *the verbal product* of a family of groups $\{A_i\}_{i \in I}$ associated with the variety $\mathcal{V}$ is defined to be

$$\mathcal{V}\prod_{i \in I} A_i = \frac{\prod_{i \in I}^{*} A_i}{V(A) \cap [A_i^A]}.$$

The verbal product is also known as varietal product or simply $\mathcal{V}$-product. If $\mathcal{V}$ is the variety of all groups, then the corresponding verbal product is the free product; if $\mathcal{V} = \mathcal{A}$ is the variety of all abelian groups, then the verbal product is the direct product and if $\mathcal{V} = \mathcal{N}_c$ is the variety of all nilpotent groups of class at most c, then the verbal product will be the nilpotent product.

Let $\{A_i \mid i \in I\}$ be a family of groups and

$$1 \to R_i \to F_i \xrightarrow{\theta_i} A_i \to 1$$

be a free presentation for A_i. We denote by θ the natural homomorphism from the free product $F = \prod_{i \in I}^{*} F_i$ onto $A = \prod_{i \in I}^{*} A_i$ induced by the θ_i. Also we assume that the group G is the verbal product of $\{A_i\}_{i \in I}$ associated with the variety $\mathcal{V}$. If ψ is the natural homomorphism from A onto G induced by the identity map on each A_i, then we have the sequence

$$F = \prod_{i \in I}^{*} F_i \xrightarrow{\theta} A = \prod_{i \in I}^{*} A_i \xrightarrow{\psi_v} G = \mathcal{V}\prod_{i \in I} A_i \to 1.$$

The following notation will be used throughout this section.

Notation 2.4.

(i) $D_1 = \prod_{i \neq j} [R_i, F_j]^F$;
(ii) $E_c = D_1 \cap \gamma_{c+1}(F)$;
(iii) $D_c = \prod_{\exists j \text{ s.t. } \mu_j \neq i} [R_i, F_{\mu_1}, \dots, F_{\mu_c}]^F$;
(iv) $K_v = V(F) \cap [F_i^F]$;
(v) $K_c = \gamma_{c+1}(F) \cap [F_i^F]$.

Let H_v be the kernel of ψ_v and R be the kernel of $\psi_v \circ \theta$. It is clear that R is actually the inverse image of H_v in F under θ, where $H_v = V(A) \cap [A_i^A]$ by the definition of the verbal product. Put $H_c = \gamma_{c+1}(A) \cap [A_i^A]$, then an immediate consequence is the following lemma.

Lemma 2.5. *With the above notation we have*

(i) $\theta(K_v) = H_v$ *and* $\theta(K_c) = H_c$;
(ii) $G = F/R$ *and* $R = \prod_{i \in I} R_i^F K_v = (\prod_{i \in I} R_i) D_1 K_v$.

Proof. (i) This follows from the definition of θ.

(ii) It is easy to see that $\ker\theta = \prod_{i\in I} R_i^F$. On the other hand, since $\theta(K_v) = \ker\psi_v$, we have $R = (\ker\theta)K_v = \prod_{i\in I} R_i^F K_v$. Also for all $r \in R_i$ and $f \in F$, $r^f = r[r, f]$. This implies that $\prod_{i\in I} R_i^F = \prod_{i\in I} R_i[R_i, F]$. Since $[R_i, F_i] \subseteq R_i$, $\prod_{i\in I} R_i^F = \prod_{i\in I} R_i D_1$. □

We now prove some lemmas to compute the c-nilpotent multiplier of G.

Lemma 2.6. *Keeping the above notation we have*

(i) $[R, {}_cF] = (\prod_{i\in I}[R_i, {}_cF_i])D_c[K_v, {}_cF]$.
(ii) *If* $V(F) \subseteq \gamma_{c+1}(F)$, *then* $R \cap \gamma_{c+1}(F) = \prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))E_c K_v$.
(iii) *If* $\gamma_{c+1}(F) \subseteq V(F)$, *then* $R \cap \gamma_{c+1}(F) = \prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))E_c K_c$.

Proof. (i)

$$\begin{aligned}
[R, {}_cF] &= \left[\prod_{i\in I} R_i^F K_v, {}_cF\right] \\
&= \prod_{i\in I}[R_i, {}_cF]^F[K_v, {}_cF] \\
&= \left(\prod_{i\in I}[R_i, {}_cF_i]\right)D_c[K_v, {}_cF].
\end{aligned}$$

(ii) Let $g \in R \cap \gamma_{c+1}(F)$. Then $g = r_{\lambda_1} \dots r_{\lambda_t} dk$ by Lemma 2.5, where $r_{\lambda_i} \in R_{\lambda_i}$, $d \in D_1$ and $k \in K_v$. Now consider the natural homomorphism

$$\varphi : F = \prod_{i\in I}^{*} F_i \to \prod_{i\in I}^{\times} F_i.$$

Since $g \in \gamma_{c+1}(F)$, $\varphi(g) = (r_{\lambda_1}, \dots, r_{\lambda_t}) \in \gamma_{c+1}(\prod_{i\in I}^{\times} F_i) = \prod_{i\in I}^{\times} \gamma_{c+1}(F_i)$. Therefore $r_{\lambda_i} \in \gamma_{c+1}(F_{\lambda_i}) \cap R_{\lambda_i}$ and then $dk \in \gamma_{c+1}(F) \cap [F_i^F]$. Now since $k \in V(F) \subseteq \gamma_{c+1}(F)$, we have $d \in \gamma_{c+1}(F) \cap D_1 = E_c$ and so the result follows.

(iii) Since $K_c \subseteq K_v$, $\prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))E_c K_c \subseteq R \cap \gamma_{c+1}(F)$. For the reverse inclusion, similar to part (i), $dk \in \gamma_{c+1}(F) \cap [F_i^F]$. Therefore $R \cap \gamma_{c+1}(F) \subseteq \prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))K_c$. Now the inclusion $E_c \subseteq K_c$ shows that the equality (iii) holds. □

Lemma 2.7. *With the above notation, let* $\varphi_c : F \to F/E_c$ *be the natural homomorphism. Then* $\varphi_c(\prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))K_v)$ *is the direct product of its subgroups* $\varphi_c(K_v)$ *and* $\varphi_c(R_i \cap \gamma_{c+1}(F_i))$, $i \in I$.

Proof. The Three Subgroups Lemma shows that

$$\bigl[R_i \cap \gamma_{c+1}(F_i), K_v\bigr] \subseteq E_c \quad \text{for all } i \in I,$$

and

$$\bigl[R_i \cap \gamma_{c+1}(F_i), R_j \cap \gamma_{c+1}(F_j)\bigr] \subseteq E_c \quad \text{for all } i, j \in I,\ i \neq j.$$

So we have

$$\bigl[\varphi_c\bigl(R_i \cap \gamma_{c+1}(F_i)\bigr), \varphi_c(K_v)\bigr] = 1 \quad \text{for all } i \in I,$$

and

$$\big[\varphi_c\big(R_i\cap\gamma_{c+1}(F_i)\big),\varphi_c\big(R_j\cap\gamma_{c+1}(F_j)\big)\big]=1\quad\text{for all } i,j\in I,\ i\neq j.$$

Moreover, by Theorem 2.2 we conclude that

$$\varphi_c\big(R_i\cap\gamma_{c+1}(F_i)\big)\cap\left(\prod_{i\neq j}\varphi_c\big(R_j\cap\gamma_{c+1}(F_j)\big)\varphi_c(K_v)\right)=1.$$

Now the result follows by the definition of the direct product. □

Lemma 2.8. *With the previous notation,*

(i) *If $V(F)\subseteq\gamma_{c+1}(F)$, then $\varphi_c(K_v)/\varphi_c([K_v,{}_cF])\cong H_v/[H_v,{}_cA]$.*
(ii) *If $\gamma_{c+1}(F)\subseteq V(F)$, then $\varphi_c(K_c)/\varphi_c([K_c,{}_cF])\cong H_c/[H_v,{}_cA]$.*

Proof. (i) If $V(F)\subseteq\gamma_{c+1}(F)$, then

$$\frac{\varphi_c(K_v)}{\varphi_c([K_v,{}_cF])}\cong\frac{K_vE_c}{[K_v,{}_cF]E_c}\cong\frac{K_v}{K_v\cap[K_v,{}_cF]E_c}.$$

On the other hand

$$\frac{\theta(K_v)}{\theta([K_v,{}_cF])}\cong\frac{K_v\ker\theta}{[K_v,{}_cF]\ker\theta}\cong\frac{K_v}{K_v\cap[K_v,{}_cF]\ker\theta}\cong\frac{K_v}{K_v\cap[K_v,{}_cF]D_1\prod_{i\in I}R_i}.$$

Now Theorem 2.2 and definition of E_c imply that

$$\frac{\theta(K_v)}{\theta([K_v,{}_cF])}\cong\frac{K_v}{K_v\cap[K_v,{}_cF]E_c}.$$

Therefore by Lemma 2.5, we conclude that

$$\frac{\varphi_c(K_v)}{\varphi_c([K_v,{}_cF])}\cong\frac{\theta(K_v)}{\theta([K_v,{}_cF])}\cong\frac{H_v}{[H_v,{}_cA]}.$$

(ii) The proof is similar to (i). □

Now we are ready to state and prove the main result of this section.

Theorem 2.9. *With the above notation,*

(i) *If $\mathcal{N}_c\subseteq\mathcal{V}$, then $\prod^{\times}_{i\in I}\mathcal{N}_cM(A_i)\times H_v/[H_v,{}_cA]$ is a homomorphic image of $\mathcal{N}_cM(\mathcal{V}\prod_{i\in I}A_i)$, and if $\mathcal{V}\prod_{i\in I}A_i$ is finite, then the above structure is isomorphic to a subgroup of $\mathcal{N}_cM(\mathcal{V}\prod_{i\in I}A_i)$.*
(ii) *If $\mathcal{V}\subseteq\mathcal{N}_c$, then $\prod^{\times}_{i\in I}\mathcal{N}_cM(A_i)\times H_c/[H_v,{}_cA]$ is a homomorphic image of $\mathcal{N}_cM(\mathcal{V}\prod_{i\in I}A_i)$, and if $\mathcal{V}\prod_{i\in I}A_i$ is finite, then the above structure is isomorphic to a subgroup of $\mathcal{N}_cM(\mathcal{V}\prod_{i\in I}A_i)$.*

Proof. (i) By Lemma 2.6(i), (ii)

$$\mathcal{N}_cM\left(\mathcal{V}\prod_{i\in I}A_i\right)\cong\frac{R\cap\gamma_{c+1}(F)}{[R,{}_cF]}\cong\frac{\prod_{i\in I}(R_i\cap\gamma_{c+1}(F_i))E_cK_v}{\prod_{i\in I}[R_i,{}_cF_i]D_c[K_v,{}_cF]}.$$

Therefore there is a natural epimorphism from $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i)$ to

$$\frac{\prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))E_c K_v}{\prod_{i\in I}[R_i,\, {}_cF_i]E_c[K_v,\, {}_cF]} \cong \frac{\varphi_c(\prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))K_v)}{\varphi_c(\prod_{i\in I}[R_i,\, {}_cF_i][K_v,\, {}_cF])}.$$

Lemma 2.7 and the fact that $\varphi_c([K_v,\, {}_cF]) \subseteq \varphi_c(K_v)$ and $\varphi_c([R_i,\, {}_cF_i]) \subseteq \varphi_c(R_i \cap \gamma_{c+1}(F_i))$ imply that

$$\frac{\varphi_c(\prod_{i\in I}(R_i \cap \gamma_{c+1}(F_i))K_v)}{\varphi_c(\prod_{i\in I}[R_i,\, {}_cF_i][K_v,\, {}_cF])} \cong \prod_{i\in I}^{\times} \frac{\varphi_c(R_i \cap \gamma_{c+1}(F_i))}{\varphi_c([R_i,\, {}_cF_i])} \times \frac{\varphi_c(K_v)}{\varphi_c([K_v,\, {}_cF])}.$$

It is straightforward to see that

$$\frac{\varphi_c(R_i \cap \gamma_{c+1}(F_i))}{\varphi_c([R_i,\, {}_cF_i])} \cong \frac{R_i \cap \gamma_{c+1}(F_i)}{[R_i,\, {}_cF_i]}$$

by Theorem 2.2. Therefore, the result holds by Lemma 2.8(i).

(ii) By an argument similar to (i), we obtain the result. □

We need the following lemma whose proof is straightforward.

Lemma 2.10. *Let $\{A_i \mid i \in I\}$ be a family of groups. Put $A = \prod_{i\in I}^{*} A_i$. Then for all integers $m \geqslant 2$,*

$$\gamma_m(A) = \prod_{i\in I} \gamma_m(A_i)\big(\gamma_m(A) \cap \big[A_i^A\big]\big).$$

In particular if the A_i are cyclic, then $\gamma_m(A) = \gamma_m(A) \cap [A_i^A]$.

The following corollary is an interesting consequence of Theorem 2.9 for cyclic groups.

Corollary 2.11. *Let $\{A_i \mid i \in I\}$ be a family of cyclic groups. Then*

(i) *If $\mathcal{N}_c \subseteq \mathcal{V}$, then $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i) \cong H_v/[H_v,\, {}_cA]$. Moreover if $\mathcal{V} \subseteq \mathcal{N}_{2c}$, then $V(\prod_{i\in I}^{\overset{2c}{*}} A_i)$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i)$.*

(ii) *If $\mathcal{V} \subseteq \mathcal{N}_c$, then $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i) \cong H_c/[H_v,\, {}_cA]$. Moreover if $\mathcal{N}_m \subseteq \mathcal{V}$, then $\gamma_{c+1}(\prod_{i\in I}^{\overset{m+c}{*}} A_i)$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i)$.*

Proof. (i) Since the A_i are cyclic groups and the R_i have no commutators, it is concluded that $D_c = E_c$. So the epimorphism in the proof of Theorem 2.9, is actually an isomorphism. Also $\mathcal{N}_c M(A_i) = 1$, therefore $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i) \cong H_v/[H_v,\, {}_cA]$. Now suppose $\mathcal{N}_c \subseteq \mathcal{V} \subseteq \mathcal{N}_{2c}$. The inclusion $V(A) \subseteq \gamma_{c+1}(A)$ and Lemma 2.10 imply that $V(A) \subseteq [A_i^A]$ and thus $H_v = V(A) \cap [A_i^A] = V(A)$. So we have $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i) = V(A)/[V(A),\, {}_cA]$ and hence $V(A)/\gamma_{2c+1}(A)$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i)$. On the other hand since $\mathcal{V} \subseteq \mathcal{N}_{2c}$, we have $V(A)/\gamma_{2c+1}(A) = V(A/\gamma_{2c+1}(A)) = V(\prod_{i\in I}^{\overset{2c}{*}} A_i)$. This completes the proof.

(ii) An argument similar to (i), shows that $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i) \cong H_c/[H_v,\, {}_cA]$. Now since $\mathcal{N}_m \subseteq \mathcal{V} \subseteq \mathcal{N}_c$, $\gamma_{c+1}(A)/\gamma_{m+c+1}(A)$ is a homomorphic image of $\mathcal{N}_c M(\mathcal{V}\prod_{i\in I} A_i)$ and also

$$\frac{\gamma_{c+1}(A)}{\gamma_{m+c+1}(A)} = \gamma_{c+1}\left(\frac{A}{\gamma_{m+c+1}(A)}\right) = \gamma_{c+1}\left(\prod_{i\in I}^{\overset{m+c}{*}} A_i\right).$$

Hence the result follows. □

Remark 2.12. Let $\{A_i \mid i \in I\}$ be a family of groups.

(i) If $\mathcal{V}$ is the variety of trivial groups, then Theorem 2.9 implies that $\prod_{i\in I}^{\times} \mathcal{N}_c M(A_i)$ is a homomorphic image of $\mathcal{N}_c M(\prod_{i\in I}^{*} A_i)$. In particular $M(\prod_{i\in I}^{*} A_i) = \prod_{i\in I}^{\times} M(A_i)$ which is a result of Miller [15].

(ii) If $\mathcal{V}$ is the variety of nilpotent groups of class at most n, $\mathcal{N}_n$, then main results of the second author [9] are obtained by Theorem 2.9 and Corollary 2.11.

3. Nilpotent products of cyclic groups

In this section we use a result of the previous section and find a structure for the c-nilpotent multiplier of the group $G = \mathbf{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbf{Z} \overset{n}{*} \mathbf{Z}_{r_1} \overset{n}{*} \cdots \overset{n}{*} \mathbf{Z}_{r_t}$, where r_{i+1} divides r_i for all i, $1 \leqslant i \leqslant t-1$, such that $(p, r_1) = 1$ for any prime p less than or equal to $n+c$. The proof relies on basic commutators [7] and related results. We recall that the number of basic commutators of weight c on n generators, denoted by $\chi_c(n)$, is determined by Witt formula [7]. Also, M. Hall proved that if F is the free group on free generators $x_1, x_2, \ldots, x_r$ and $c_1, \ldots, c_t$ are basic commutators of weight $1, 2, \ldots, n$, on $x_1, \ldots, x_r$, then an arbitrary element f of F has a unique representation,

$$f = c_1^{\beta_1} c_2^{\beta_2} \ldots c_t^{\beta_t} \mod \gamma_{n+1}(F).$$

In particular the basic commutators of weight n provide a basis for the free abelian group $\gamma_n(F)/\gamma_{n+1}(F)$ (see [7]).

The following theorem represents the elements of some nilpotent products of cyclic groups in terms of basic commutators.

Theorem 3.1. *(See [18].) Let $A_1, \ldots, A_t$ be cyclic groups of order $\alpha_1, \ldots, \alpha_t$ respectively, where if A_i is infinite cyclic, then $\alpha_i = 0$. Let a_i generate A_i and let $G = A_1 \overset{n}{*} \cdots \overset{n}{*} A_t$, where n is greater than or equal to 2. Suppose that all the primes appearing in the factorizations of the α_i are greater than or equal to n and $u_1, u_2, \ldots,$ are basic commutators of weight less than n, on the letters $a_1, \ldots, a_t$. Put $N_i = \alpha_{i_j}$ if $u_i = a_{i_j}$ of weight 1, and*

$$N_i = gcd(\alpha_{i_1}, \ldots, \alpha_{i_k})$$

if a_{i_j}, $1 \leqslant j \leqslant k$, appears in u_i. Then every element g of G can be uniquely expressed as

$$g = \prod u_i^{m_i},$$

where the m_i are integers modulo N_i (by gcd we mean the greatest common divisor).

The following theorem is an interesting consequence of Corollary 2.11.

Theorem 3.2. *Let $\{A_i \mid i \in I\}$ be a family of cyclic groups. Then*

(i) *if $n \geqslant c$, then $\mathcal{N}_c M(\prod_{i\in I}^{\overset{n}{*}} A_i) \cong \gamma_{n+1}(\prod_{i\in I}^{\overset{n+c}{*}} A_i)$;*

(ii) *if $c \geqslant n$, then $\mathcal{N}_c M(\prod_{i\in I}^{\overset{n}{*}} A_i) \cong \gamma_{c+1}(\prod_{i\in I}^{\overset{n+c}{*}} A_i)$.*

Proof. (i) Put $\mathcal{V} = \mathcal{N}_n$ in Corollary 2.11 and deduce that

$$\mathcal{N}_c M\left(\prod_{i\in I}^{\overset{n}{*}} A_i\right) \cong H_n/[H_n, {}_c A].$$

On the other hand by Lemma 2.10, $H_n = \gamma_{n+1}(A) \cap [A_i^A] = \gamma_{n+1}(A)$. Therefore

$$\mathcal{N}_c M\left(\prod_{i\in I}^{\overset{n}{*}} A_i\right) \cong \frac{\gamma_{n+1}(A)}{[\gamma_{n+1}(A),\ {}_cA]} = \gamma_{n+1}\left(\frac{A}{\gamma_{n+c+1}(A)}\right) = \gamma_{n+1}\left(\prod_{i\in I}^{\overset{n+c}{*}} A_i\right).$$

(ii) The result follows as for (i). □

Now, we are in a position to state and prove the main result of this section.

Theorem 3.3. *Let $G = A_1 \overset{n}{*} \cdots \overset{n}{*} A_{m+t}$ be the nth nilpotent product of cyclic groups such that $A_i \cong \mathbf{Z}$ for $1 \leqslant i \leqslant m$ and $A_{m+j} \cong \mathbf{Z}_{r_j}$ and $r_{j+1} \mid r_j$ for all $1 \leqslant j \leqslant t-1$. If $(p, r_1) = 1$ for any prime p less than or equal to $n+c$, then*

(i) *if $n \geqslant c$, then $\mathcal{N}_c M(G) \cong \mathbf{Z}^{(g_0)} \oplus \mathbf{Z}_{r_1}^{(g_1-g_0)} \oplus \cdots \oplus \mathbf{Z}_{r_t}^{(g_t-g_{t-1})}$;*
(ii) *if $c \geqslant n$, then $\mathcal{N}_c M(G) \cong \mathbf{Z}^{(f_0)} \oplus \mathbf{Z}_{r_1}^{(f_1-f_0)} \oplus \cdots \oplus \mathbf{Z}_{r_t}^{(f_t-f_{t-1})}$,*

where $f_k = \sum_{i=1}^{n} \chi_{c+i}(m+k)$ and $g_k = \sum_{i=1}^{c} \chi_{n+i}(m+k)$ for $0 \leqslant k \leqslant t$ and $\mathbf{Z}_r^{(d)}$ denotes the direct sum of d copies of the cyclic group $\mathbf{Z}_r$.

Proof. (i) If $n \geqslant c$, then by Theorem 3.2, it is enough to find the structure of $\gamma_{n+1}(\prod_{i\in I}^{\overset{n+c}{*}} A_i)$. Suppose that $A_i = \langle a_i \mid a_i^{\alpha_i}\rangle$ for $1 \leqslant i \leqslant m+t$ such that $\alpha_i = 0$ for $1 \leqslant i \leqslant m$ and $\alpha_{m+i} = r_i$ for $1 \leqslant i \leqslant t$. Also let F be the free group generated by $a_1, \ldots, a_{m+t}$ and B be the set of all basic commutators of weight $1, 2, \ldots, c+n$ on the letters $a_1, \ldots, a_{m+t}$. Now define

$$D = \left\{u^{N_i} \;\middle|\; u \in B \text{ and } N_i = gcd(\alpha_{i_1}, \ldots, \alpha_{i_k}) \text{ if } a_{i_j} \text{ appears in } u \text{ for } 1 \leqslant j \leqslant k\right\}.$$

Then Theorem 3.1 implies that $\prod_{i\in I}^{\overset{n+c}{*}} A_i = F/\langle D\rangle\gamma_{c+n+1}(F)$ and so

$$\begin{aligned}\gamma_{n+1}\left(\prod_{i\in I}^{\overset{n+c}{*}} A_i\right) &= \gamma_{n+1}\left(\frac{F}{\langle D\rangle\gamma_{c+n+1}(F)}\right)\\ &= \frac{\gamma_{n+1}(F)}{\langle D\rangle\gamma_{c+n+1}(F) \cap \gamma_{n+1}(F)}\\ &\cong \frac{\gamma_{n+1}(F)/\gamma_{c+n+1}(F)}{(\langle D\rangle \cap \gamma_{n+1}(F))\gamma_{c+n+1}(F)/\gamma_{c+n+1}(F)}.\end{aligned}$$

It can be deduced from Hall Theorem that $\gamma_{n+1}(F)/\gamma_{c+n+1}(F)$ is a free abelian group with a basis $\bar{B}_1 = \{u\gamma_{c+n+1}(F) \mid\ u \in B_1\}$, where B_1 is the set of all basic commutators of weight $n+1, \ldots, c+n$ on $a_1, \ldots, a_{m+t}$. Also, the uniqueness of the presentation of elements implies that the abelian group $(\langle D\rangle \cap \gamma_{n+1}(F))\gamma_{c+n+1}(F)/\gamma_{c+n+1}(F)$ is free with a basis

$$\bar{E} = \left\{u\gamma_{c+n+1}(F) \;\middle|\; u \in \bigcup_{j=1}^{t} D_j\right\},$$

where D_j is the set of all u^{r_j}, such that u is a basic commutator of weight $n+1, \dots, c+n$ on $a_1, \dots, a_{m+j}$ such that a_{m+j} appears in u. Also we have

$$|D_j| = \sum_{i=1}^{c} \chi_{n+i}(m+j) - \chi_{n+i}(m+j-1) = g_j - g_{j-1}.$$

This completes the proof.

(ii) The proof is similar to (i). □

Note that the authors with F. Mohammadzadeh [8] by a different method presented a similar structure for $\mathcal{N}_c M(G)$, for $c \geqslant n$ with a weaker condition $(p, r_1) = 1$ for any prime p less than or equal to n.

Remark 3.4. The condition $r_{j+1} \mid r_j$, in the above theorem, simplifies the structure of the c-nilpotent multiplier of G and gives a clear formula. One can use the above method and find the structure of $\mathcal{N}_c M(G)$ without the condition $r_{j+1} \mid r_j$, but with a more complex formula. For example, for a simple case if $G = \mathbf{Z}_r \overset{n}{*} \mathbf{Z}_s$ where $(p, r) = (p, s) = 1$ for any prime p less than or equal to $n+c$ and $(r, s) = d$, then

(i) if $n \geqslant c$, then $\mathcal{N}_c M(G) \cong \mathbf{Z}_d^{(\sum_{i=1}^{c} \chi_{n+i}(2))}$;
(ii) if $c \geqslant n$, then $\mathcal{N}_c M(G) \cong \mathbf{Z}_d^{(\sum_{i=1}^{n} \chi_{c+i}(2))}$.

References

[1] F.R. Beyl, J. Tappe, Group Extensions, Representations and the Schur Multiplicator, Lecture Notes in Math., vol. 958, Springer-Verlag, Berlin, 1982.
[2] G. Ellis, On groups with a finite nilpotent upper central quotient, Arch. Math. 70 (1998) 89–96.
[3] G. Ellis, On the relation between upper central quotients and lower central series of a group, Trans. Amer. Math. Soc. 353 (2001) 4219–4234.
[4] O.N. Golovin, Nilpotent products of groups, in: Amer. Math. Soc. Transl. Ser. 2, vol. 2, 1956, pp. 89–115.
[5] N.D. Gupta, M.R.R. Moghaddam, Higher Schur multiplicators of nilpotent dihedral groups, C. R. Math. Rep. Acad. Sci. Canada XIV (5) (1992) 225–230.
[6] H. Haebich, The multiplicator of a regular product of groups, Bull. Austral. Math. Soc. 7 (1972) 279–296.
[7] M. Hall, The Theory of Groups, The Macmillan Company, New York, 1959.
[8] A. Hokmabadi, B. Mashayekhy, F. Mohammadzadeh, Polynilpotent multipliers of some nilpotent products of cyclic groups II, Int. J. Math. Game Theory Algebra, in press.
[9] B. Mashayekhy, Some notes on the Baer-invariant of a nilpotent product of groups, J. Algebra 235 (2001) 15–26.
[10] B. Mashayekhy, M.R.R. Moghaddam, Higher Schur multiplicator of a finite abelian group, Algebra Colloq. 4 (3) (1997) 317–322.
[11] B. Mashayekhy, M. Parvizi, On polynilpotent multipliers of free nilpotent groups, Comm. Algebra 34 (6) (2006) 2287–2294.
[12] B. Mashayekhy, M. Parvizi, Some Baer invariants of free nilpotent groups, J. Algebra 317 (2007) 365–375.
[13] M.R.R. Moghaddam, The Baer-invariant of a direct product, Arch. Math. 33 (1979) 504–511.
[14] M.R.R. Moghaddam, B. Mashayekhy, S. Kayvanfar, The higher Schur multiplicator of certain class of groups, Southeast Asian Bull. Math. 27 (2003) 121–128.
[15] C. Miller, The second homology group of a group: Relations among commutators, Proc. Amer. Math. Soc. 3 (1952) 588–595.
[16] H. Neumann, Varieties of Groups, Springer-Verlag, Berlin, 1967.
[17] I. Schur, Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, J. Math. 132 (1907) 85–137.
[18] R.R. Struik, On nilpotent products of cyclic groups, Canad. J. Math. 12 (1960) 447–462.
[19] J. Wiegold, The multiplicator of a direct product, Q. J. Math. (2) 22 (1971) 103–105.