WEYL-HEISENBERG FRAMES AND φ-FACTORABLE OPERATORS ON LCA GROUPS

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ABSTRACT. This paper is an investigation of factorable operators, Riesz bases and Weyl-Heisenberg frames on G with respect to a function-valued inner product, the so called φ-bracket product, on $L^2(G)$ where G is a locally compact abelian group and φ is a topological isomorphism on G. We investigate φ-factorable operators on $L^2(G)$ and extend Riesz Representation Theorems for these operators. Also we introduce φ-Riesz bases and obtain a characterization of them in $L^2(G)$. Finally we show that several well known theorems for Weyl-Heisenberg frames in $L^2(\mathbb{R})$ remain valid in $L^2(G)$, and they are unified under the aspects of group theory, in connection with the φ-bracket product.

1. INTRODUCTION AND PRELIMINARIES

In [12] we have defined the φ-bracket product as a function-valued inner product on $L^2(G)$, where G is a locally compact abelian (LCA) group and φ is a topological isomorphism on G. The φ-bracket product as a new inner product on $L^2(G)$ is applicable to extend many ideas and constructions from the theory of factorable operators and Weyl-Heisenberg frames on $\mathbb{R}^n$, to the setting of LCA groups in a more general and different way. Utilizing factorable operators we can define and investigate Riesz bases in this new setting. Whereas [12] was devoted to basic properties of the φ-bracket product and φ-bases, this paper deals with characterizing φ-Riesz bases and φ-factorable operators on $L^2(G)$. We continue our investigation indepedently of [12], following the line of approach worked by Casazza and Lammers [4], but in a more general setting, using various tools in abstract harmonic analysis. We define and investigate φ-factorable operators and φ-Riesz bases in $L^2(G)$. We then study Weyl-Heisenberg frames on LCA groups, in connection with the φ-bracket product. Our results generalize some of the results appearing in the literature on Riesz bases and Weyl-Heisenberg frames. Such a unified approach is useful, since it determines the basic features of them, and includes most of the special cases. Also it leads to the interesting question of how to formulate related topics (such as frames) in the LCA group setting.

Here we give some of the basics regarding LCA groups. For a comprehensive account of LCA groups we refer to [7, 10]. Suppose G is a LCA group with the Haar measure dx. A subgroup L of G is called a *uniform lattice* if it is discrete and co-compact (i.e G/L is compact). Let φ be a topological isomorphism on G. If L is a uniform lattice in G, then so is $\varphi(L)$. Indeed, obviously $\varphi(L)$ is discrete. Also by [10, Theorem 5.34] $G/\varphi(L)$ is topologically isomorphic to G/L and so it is compact. In

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this paper we always assume that $G/\varphi(L)$ is normalized i.e. $|G/\varphi(L)| = 1$. Denote by $\varphi(L)^\perp$ the annihilator of $\varphi(L)$ in $\hat{G}$, i.e. $\varphi(L)^\perp = \{\gamma \in \hat{G};\ \gamma(\varphi(L)) = \{1\}\}$, which is a uniform lattice in $\hat{G}$ (see [13, 14, 15]).

Let L be a uniform lattice in G. Choosing the counting measure on L, a relation between the Haar measures dx on G and $d\dot{x}$ on $G/\varphi(L)$ is given by the following special case of Weil's formula [7]:
For $f \in L^1(G)$, we have $\sum_{k\in L} f(x\varphi(k^{-1})) \in L^1(G/\varphi(L))$ and

$$\int_G f(x)dx = \int_{G/\varphi(L)} \sum_{\varphi(k^{-1})\in\varphi(L)} f(x\varphi(k^{-1}))d\dot{x}, \tag{1.1}$$

where $\dot{x} = x\varphi(L)$.

Let $f, g \in L^2(G)$. The *φ-bracket product* of f, g is defined by

$$[f,g]_\varphi(\dot{x}) = \sum_{k\in L} f\overline{g}(x\varphi(k^{-1})), \tag{1.2}$$

for all $x \in G$. We define the φ-norm of f as $||f||_\varphi(\dot{x}) = ([f,f]_\varphi(\dot{x}))^{1/2}$. In the sequel we collect several basic properties of the φ-bracket product which follow by easy direct computations. The reader who does not want to check the details is referred to [12]. Let $f,\ g \in L^2(G)$. Then $|[f,g]_\varphi| \leq ||f||_\varphi ||g||_\varphi$ (Cauchy Schwartz Inequality). Also obviously (1.1) implies $\int_{G/\varphi(L)}[f,g]_\varphi(\dot{x})d\dot{x} = < f, g >_{L^2(G)}$. For $\gamma \in \hat{G}$, denote by M_γ the modulation operator on $L^2(G)$, i.e. $M_\gamma f(x) = \gamma(x)f(x)$, for all $f \in L^2(G)$. Then for $f, g \in L^2(G)$ and $\gamma \in \varphi(L)^\perp$ we have the following relation between the φ-bracket product and the usual inner product in $L^2(G)$:

$$\widehat{[f,g]_\varphi}(\gamma) = < f, M_\gamma g >_{L^2(G)} . \tag{1.3}$$

We say $g \in L^2(G)$ is *φ-bounded* if there exists $M > 0$ so that $||g||_\varphi \leq M$ *a.e.*. For $f, g \in L^2(G)$ the function $[f,g]_\varphi g$ need not generally be in $L^2(G)$. But we have

Proposition 1.1. *If $f, g, h \in L^2(G)$ and g, h are φ-bounded then $[f,g]_\varphi h \in L^2(G)$.*

A sequence $(g_n)_{n\in\mathbb{N}} \subseteq L^2(G)$ is called *φ-orthonormal* if $[g_n, g_m]_\varphi = 0$, for all $n \neq m \in \mathbb{N}$ and $||g_n||_\varphi = 1$ for all $n \in \mathbb{N}$. Let $f \in L^2(G)$ and $(g_n)_{n\in\mathbb{N}}$ be a φ-orthonormal sequence in $L^2(G)$. An extension of [4, Theorem 4.13] from $\mathbb{R}$ to the setting of a LCA group gives Bessel's Inequality for φ-bracket products as follows:

$$\sum_{n\in\mathbb{N}} |[f,g_n]_\varphi(\dot{x})|^2 \leq ||f||^2_\varphi(\dot{x}), \quad \textit{for a.e. } \dot{x} \in G/\varphi(L). \tag{1.4}$$

A φ-orthonormal sequence $(g_n)_{n\in\mathbb{N}}$ is called a *φ-orthonormal basis* if $[f,g_n]_\varphi = 0$ a.e. for all $n \in \mathbb{N}$, implies $f = 0$ a.e. Let $(g_n)_{n\in\mathbb{N}}$ be a φ-orthonormal sequence. It is not difficult to mimic the standard proofs for a usual orthonormal sequence in a Hilbert space to obtain equivalent conditions for $(g_n)_{n\in\mathbb{N}} \subseteq L^2(G)$ to be a φ-orthonormal basis.

Proposition 1.2. *If $(g_n)_{n\in\mathbb{N}}$ is a φ-orthonormal sequence in $L^2(G)$, the following are equivalent.*

1. *$(g_n)_{n\in\mathbb{N}}$ is a maximal φ-orthonormal sequence, i.e. $(g_n)_{n\in\mathbb{N}}$ is not contained in any other φ-orthonormal set.*
2. *$(g_n)_{n\in\mathbb{N}}$ is a φ-orthonormal basis.*
3. *For each $f \in L^2(G)$, $f = \sum_{n\in\mathbb{N}}[f,g_n]_\varphi g_n$ a.e.*

(4) $\|f\|_\varphi^2 = \sum_{n\in\mathbb{N}} |[f, g_n]_\varphi|^2$ *a.e. for all* $f \in L^2(G)$ *(Parseval Identity).*
(5) $\{M_\gamma g_n\}_{n\in\mathbb{N},\gamma\in\varphi(L)^\perp}$ *is an orthonormal basis for* $L^2(G)$.

Thanks to Zorn's Lemma and Proposition 1.2, $L^2(G)$ admits a φ-orthonormal basis.

The rest of this paper is organized as follows. In Section 2 we introduce a φ-factorable operator as an extension of the analogous one in [4]. We show that most of the authors' results on $\mathbb{R}$, especially the Riesz Representation Theorems remain valid for a LCA group. In Section 3 we define a φ-Riesz basis in $L^2(G)$ and establish equivalent conditions for a sequence to be a φ-Riesz basis.

Over the last ten years, there have been a lot of researches about frame theory in general and Weyl-Heisenberg frame theory in particular. While most of them are on the Euclidian space, just a few generalizations to LCA groups have been presented [14, 16, 11, 6]. Our main goal in Section 4 is to represent Weyl-Heisenberg frame identity and the frame operator of a Weyl-Heisenberg frame in terms of the φ-bracket product and to extend some of the results in this theory from $\mathbb{R}$ to an LCA group.

2. φ-Factorable Operators

Throughout this paper we always assume that G is a second countable LCA group, φ is a topological isomorphism on G and the notation are as in Section 2.

A function $h \in L^\infty(G)$ is said to be *φ-periodic* if $h(x\varphi(k)) = h(x)$ for every $k \in L$, $x \in G$.

Definition 2.1. We say an operator $U : L^2(G) \to L^p(E)$, $1 \le p \le \infty$, is *φ-factorable* if $U(hf) = hU(f)$ for all $f \in L^2(G)$ and all φ-periodic $h \in L^\infty(G)$, where E is a subgroup of G or $G/\varphi(L)$.

A bounded operator U is φ-factorable if and only if it commutes with modulations. More precisely:

Lemma 2.2. *Let U be a bounded operator from $L^2(G)$ to $L^2(E)$, where E is a subgroup of G or $G/\varphi(L)$. U is φ-factorable if and only if*

$$U(M_\gamma g) = M_\gamma U(g) \text{ for all } g \in L^2(G),\ \gamma \in \varphi(L)^\perp. \tag{2.1}$$

Proof. If U is φ-factorable and $\gamma \in \varphi(L)^\perp (\subseteq \hat{G} \subseteq L^\infty(G))$ then since γ is φ-periodic, (2.1) obviously holds. Conversely, assume (2.1). Then U is φ-factorable using the facts that $\varphi(L)^\perp (= \widehat{G/\varphi(L)})$ is an orthonormal basis for $L^2(G/\varphi(L))$ and $L^\infty(G/\varphi(L)) \subseteq L^2(G/\varphi(L))$. Note that there is a one-to-one correspondence between $L^\infty(G/\varphi(L))$ and the set of all φ-periodic $h \in L^\infty(G)$. □

Our main goal in this section is to characterize φ-factorable operators $U : L^2(G) \to L^p(G/\varphi(L))$, for $p = 1$ and $p = 2$.

Clearly the operator U defined by $U(f) = [f, g]_\varphi$ for $f \in L^2(G)$, is φ-factorable. We will also show that every φ-factorable operator $U : L^2(G) \to L^1(G/\varphi(L))$ is of this form. First we establish a lemma in which we show that two φ-factorable operators are equal on $L^2(G)$ if and only if their integrals over $G/\varphi(L)$ are the same.

Lemma 2.3. *Let $U_1, U_2 : L^2(G) \to L^1(G/\varphi(L))$ be two φ-factorable operators. Then $U_1 = U_2$ if and only if $\int_{G/\varphi(L)} U_1(f)(\dot{x})d\dot{x} = \int_{G/\varphi(L)} U_2(f)(\dot{x})d\dot{x}$, for every $f \in L^2(G)$.*

Proof. The necessity is obvious. For the converse, by [7, Theorem 4.33] it is enough to show that $\widehat{U_1(f)} = \widehat{U_2(f)}$ for all $f \in L^2(G)$. Let $\xi \in \varphi(L)^{\perp}$ and $f \in L^2(G)$. Since ξ as a function in $L^{\infty}(G)$ is φ-periodic we obtain

$$\begin{aligned} \widehat{U_1(f)}(\xi) &= \int_{G/\varphi(L)} U_1(f)(\dot{x})\bar{\xi}(\dot{x})d\dot{x} \\ &= \int_{G/\varphi(L)} U_1(\xi^{-1}.f)(\dot{x})d\dot{x} \\ &= \int_{G/\varphi(L)} U_2(\xi^{-1}.f)(\dot{x})d\dot{x} \\ &= \widehat{U_2(f)}(\xi). \end{aligned}$$

Hence $U_1 = U_2$. □

Now we have the following Riesz Representation Theorem which characterizes all φ-factorable operators from $L^2(G)$ to $L^1(G/\varphi(L))$.

Theorem 2.4. *A bounded operator $U : L^2(G) \to L^1(G/\varphi(L))$ is φ-factorable if and only if there exists $g \in L^2(G)$ such that $U(f) = [f, g]_{\varphi}$ a.e. for all $f \in L^2(G)$. Moreover $||U|| = ||g||$.*

Proof. Let $U : L^2(G) \to L^1(G/\varphi(L))$ be a bounded φ-factorable operator. Define the linear functional $\psi : L^2(G) \to \mathbb{C}$ by $\psi(f) = \int_{G/\varphi(L)} U(f)(\dot{x})d\dot{x}$. By the standard Riesz Representation Theorem [8, Theorem 5.25], there exists $g \in L^2(G)$ such that $\psi(f) =< f, g >_{L^2(G)}$ for all $f \in L^2(G)$. Thus $\int_{G/\varphi(L)} U(f)(\dot{x})d\dot{x} = \psi(f) =< f, g >_{L^2(G)} = \int_{G/\varphi(L)} [f, g]_{\varphi}(\dot{x})d\dot{x}$. By Lemma 2.3, $U(f) = [f, g]_{\varphi}$ a.e. for all $f \in L^2(G)$. Moreover, for any $f \in L^2(G)$,

$$\begin{aligned} ||U(f)||_1 &= \int_{G/\varphi(L)} |[f, g]_{\varphi}(\dot{x})|d\dot{x} \\ &\leq \int_{G/\varphi(L)} ||f||_{\varphi}(\dot{x})||g||_{\varphi}(\dot{x})d\dot{x} \\ &\leq (\int_{G/\varphi(L)} ||f||^2_{\varphi}(\dot{x})d\dot{x})^{1/2}(\int_{G/\varphi(L)} ||g||^2_{\varphi}(\dot{x})d\dot{x})^{1/2} \\ &= ||f||_2||g||_2. \end{aligned}$$

So $||U|| \leq ||g||_2$. Also $||Ug||_1 = \int_{G/\varphi(L)} |[g, g]_{\varphi}(\dot{x})|d\dot{x} = ||g||_2^2$. Therefore $||U|| = ||g||_2$. □

The following theorem characterizes φ-factorable operators from $L^2(G)$ to $L^2(G/\varphi(L))$.

Theorem 2.5. *A bounded operator $U : L^2(G) \to L^2(G/\varphi(L))$ is φ-factorable if and only if there exists a φ-bounded $g \in L^2(G)$ such that $U(f) = [f, g]_{\varphi}$ a.e. for all $f \in L^2(G)$. Moreover $||U||^2 = ess\ sup_{\dot{x}\in G/\varphi(L)}||g||^2_{\varphi}(\dot{x})$.*

Proof. Let $U(f) = [f, g]_{\varphi}$ *a.e.* for some φ-bounded $g \in L^2(G)$. Then obviously U is φ-factorable and by Cauchy Shwartz Inequality we have

$$\begin{aligned} ||U(f)||^2_{L^2(G/\varphi(L))} &= \int_{G/\varphi(L)} |U(f)(\dot{x})|^2 d\dot{x} \\ &= \int_{G/\varphi(L)} |[f, g]_{\varphi}(\dot{x})|^2 d\dot{x} \\ &\leq \int_{G/\varphi(L)} ||f||^2_{\varphi}(\dot{x})||g||^2_{\varphi}(\dot{x})d\dot{x} \\ &\leq ess\ sup_{\dot{x}\in G/\varphi(L)}||g||^2_{\varphi}(\dot{x})||f||^2_{L^2(G)}. \end{aligned} \tag{2.2}$$

Letting $f = g$ above we get $||U|| = ess\ sup_{\dot{x}\in G/\varphi(L)}||g||_{\varphi}(\dot{x})$.

For the converse, let U be a φ-factorable operator from $L^2(G)$ to $L^2(G/\varphi(L))$. Since $G/\varphi(L)$ is compact, $L^2(G/\varphi(L)) \subseteq L^1(G/\varphi(L))$ and so by Theorem 2.4, there exists $g \in L^2(G)$ such that $U(f) = [f, g]_{\varphi}$ *a.e.* for all $f \in L^2(G)$. But also g is

φ-bounded. To show this observe that $|U(g)(\dot{x})| \le ||U|| ||g||_\varphi(\dot{x})$ for a.e. $\dot{x} \in G/\varphi(L)$. In fact, for every φ-periodic $h \in L^\infty(G)$ we have

$$\begin{aligned}
\int_{G/\varphi(L)} |h(\dot{x})|^2 |U(g)(\dot{x})|^2 d\dot{x} &= \int_{G/\varphi(L)} |U(hg)(\dot{x})|^2 d\dot{x} \\
&= ||U(hg)||^2_{L^2(G/\varphi(L))} \\
&\le ||U||^2 \int_G |hg(x)|^2 dx \\
&= ||U||^2 \int_{G/\varphi(L)} \sum_{\varphi(k)\in\varphi(L)} |hg(x\varphi(k^{-1}))|^2 d\dot{x} \\
&= ||U||^2 \int_{G/\varphi(L)} |h(\dot{x})|^2 \sum_{\varphi(k)\in\varphi(L)} |g(x\varphi(k^{-1}))|^2 d\dot{x} \\
&= ||U||^2 \int_{G/\varphi(L)} |h(\dot{x})|^2 ||g||^2_\varphi(\dot{x}) d\dot{x},
\end{aligned}$$

that is $|U(g)(\dot{x})| \le ||U|| ||g||_\varphi(\dot{x})$ for a.e. $\dot{x} \in G/\varphi(L)$. So we get $||g||^2_\varphi(\dot{x}) = |U(g)(\dot{x})| \le ||U|| ||g||_\varphi(\dot{x})$ for a.e. $\dot{x} \in G/\varphi(L)$. Hence $||g||_\varphi(\dot{x}) \le ||U||$ *a.e.* That is g is φ-bounded. □

Next we show that every bounded φ-factorable operator on $L^2(G)$ is adjointable.

Proposition 2.6. *Let $U : L^2(G) \to L^2(G)$ be a bounded φ-factorable operator and U^* be its adjoint. Then U^* is φ-factorable. Moreover,*

$$[U(f), g]_\varphi = [f, U^*(g)]_\varphi, \quad \textit{a.e. for all } f, g \in L^2(G). \tag{2.3}$$

Proof. Clearly U^* is φ-factorable. Indeed, for $f, g \in L^2(G)$ and φ-periodic $h \in L^\infty(G)$ we have

$$\begin{aligned}
< U^*(hf), g >_{L^2(G)} &= < hf, U(g) >_{L^2(G)} \\
&= < f, \bar{h}U(g) >_{L^2(G)} \\
&= < f, U(\bar{h}g) >_{L^2(G)} \\
&= < U^*(f), \bar{h}g >_{L^2(G)} \\
&= < hU^*(f), g >_{L^2(G)} .
\end{aligned}$$

Moreover, given $f, g \in L^2(G)$ we have

$$\begin{aligned}
\int_{G/\varphi(L)} [U(f), g]_\varphi(\dot{x}) d\dot{x} &= < U(f), g >_{L^2(G)} \\
&= < f, U^*(g) >_{L^2(G)} \\
&= \int_{G/\varphi(L)} [f, U^*(g)]_\varphi(\dot{x}) d\dot{x},
\end{aligned}$$

which implies (2.3). □

Example 2.7. For applications the most important class of LCA groups is the class of compactly generated LCA Lie groups. By the Structure Theorem for compactly generated LCA Lie groups, these groups are of the form $\mathbb{R}^p \times \mathbb{Z}^q \times \mathbb{T}^r \times F$, where $p, q, r \in \mathbb{N}_0$ and F is a finite abelian group (see [10]). Let $G = \mathbb{R}^n \times \mathbb{Z}^n \times \mathbb{T}^n \times \mathbb{Z}_n$ for $n \in \mathbb{N}$, where $\mathbb{Z}_n$ is the finite abelian group $\{0, 1, 2, ..., n-1\}$ of residues modulo n. Then $L = \mathbb{Z}^n \times \mathbb{Z}^n \times \{1\} \times \mathbb{Z}_n$ is a uniform lattice in G. Let A be an invertible $n \times n$ real matrix and fix $l \in \mathbb{Z}^n$. Define $\varphi : G \to G$ by $\varphi(x, m, t, p) = (Ax, l + m, t, p)$, for every $x \in \mathbb{R}^n$, $m \in \mathbb{Z}^n$, $t \in \mathbb{T}^n$, $p \in \mathbb{Z}_n$. Then for $g \in L^2(G)$, the operator U given by $U(f) = [f, g]_\varphi$, where $[f, g]_\varphi(x, m, t, p) = \sum_{k\in\mathbb{Z}^n, n\in\mathbb{Z}^n, q\in\mathbb{Z}_n} f\bar{g}(x - Ak, m - l + n, t - 1, p - q)$, is a φ-factorable operator from $L^2(G)$ to $L^1(G/\varphi(L))$ $(= L^1(\mathbb{T}^n \times \{1\} \times \mathbb{T}^n \times \{1\})$.

Example 2.8. Fix a prime p. Let Δ_p denote the group of p-adic integers as defined in [10, Definition 10.2]. Consider the LCA group $G = \mathbb{R} \times \Delta_p$ and let L be the subgroup $\{(n, n\mathbf{u})\}_{n\in\mathbb{Z}}$ of $\mathbb{R} \times \Delta_p$, where $\mathbf{u} = (1, 0, 0, ...)$. Then L is a uniform lattice in $\mathbb{R} \times \Delta_p$ (obviously L is discrete and by [10, Theorem 10.13], $\mathbb{R} \times \Delta_p/L$ is compact).

Let $\mathbf{a} := (1/p, 0, 0, ...) \in \Delta_p$. Then the mapping $\varphi : \mathbb{R} \times \Delta_p \to \mathbb{R} \times \Delta_p$ defined for $(x, \mathbf{v}) \in \mathbb{R} \times \Delta_p$, by $\varphi(x, \mathbf{v}) = (2x, \mathbf{av})$, is a topological automorphism on $\mathbb{R} \times \Delta_p$. For $g \in L^2(\mathbb{R} \times \Delta_p)$, the operator U given by $U(f)(x, \mathbf{v}) = \sum_{k\in\mathbb{Z}} f\bar{g}(x - 2k, \mathbf{v} - k\mathbf{au})$ is a φ-factorable operator from $L^2(\mathbb{R} \times \Delta_p)$ to $L^1(\mathbb{R} \times \Delta_p/L)$.

Example 2.9. Consider the subgroup of the upper triangular 3×3-matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}$, where $a, b \in \mathbb{R}$. This group can be identified with $\mathbb{R}^2$ equipped with the product $(a_1, b_1) \circ (a_2, b_2) = (a_1 + a_2, b_1 + b_2 - a_1 a_2)$. Then $G = (\mathbb{R}^2, \circ)$ is an LCA group with the identity $e = (0, 0)$ and $(a, b)^{-1} = (-a, -b - a^2)$. Clearly $L = \mathbb{Z}^2$ is a uniform lattice in G. The mapping $\varphi : G \to G$ defined by $\varphi(a, b) = (2a, 4b)$, is a topological isomorphism on G. For $g \in L^2(G)$, the operator U given by $U(f)(a, b) = \sum_{(k,l)\in\mathbb{Z}^2} f\bar{g}(a - 2k, b - 4l - 4k^2 + 2ka)$ is a φ-factorable operator.

Our goal in the next section is to define and investigate φ-Riesz bases in $L^2(G)$, applying φ-factorable operators.

3. φ-Riesz Bases in $L^2(G)$

Riesz bases in $L^2(\mathbb{R})$ have several equivalent definitions (see [5, 9, 17]). This section sets out equivalent conditions for a sequence in $L^2(G)$ to be a φ-Riesz basis, where G is a second countable LCA group and φ is a topological isomorphism on G. We start with the definition.

Definition 3.1. A sequence $(f_n)_{n\in\mathbb{N}}$ in $L^2(G)$ is said to be a *φ-Riesz basis* if there exists a φ-orthonormal basis $(g_n)_{n\in\mathbb{N}}$ and a φ-factorable operator $U : L^2(G) \to L^2(G)$, which is a topological isomorphism and $U(g_n) = f_n$, for every $n \in \mathbb{N}$.

We introduce a φ-complete (φ-total) sequence in $L^2(G)$ as follows:

Definition 3.2. Given a sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^2(G)$, by $\overline{span}^{\|\cdot\|_\varphi}(f_n) = L^2(G)$ we mean that for every $f \in L^2(G)$ there exists a sequence $\{h_n\}_{n\in\mathbb{N}} \subseteq L^\infty(G/\varphi(L))$ with $\sum_{n=1}^\infty |h_n(\dot{x})|^2 < \infty$ a.e., such that $f = \sum_{n=1}^\infty h_n f_n$, a.e. We say a sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^2(G)$ is *φ-complete (φ-total)* in $L^2(G)$, if $\overline{span}^{\|\cdot\|_\varphi}(f_n) = L^2(G)$.

The following lemma will be needed in the proof of Theorem 3.4.

Lemma 3.3. *Suppose U is a bounded φ-factorable operator on $L^2(G)$. For every $f \in L^2(G)$, we have $\|Uf\|_\varphi \le \|U\|\|f\|_\varphi$ a.e.*

Proof. For every φ-periodic $h \in L^\infty(G)$, we have

$$\begin{aligned}
\int_{G/\varphi(L)} |h(\dot{x})|^2 \|U(f)\|_\varphi^2(\dot{x})d\dot{x} &= \int_{G/\varphi(L)} \sum_{k\in L} |U(f)(x\varphi(k^{-1}))|^2 |h(x\varphi(k^{-1}))|^2 d\dot{x} \\
&= \int_{G/\varphi(L)} \sum_{k\in L} |U(hf)(x\varphi(k^{-1}))|^2 d\dot{x} \\
&= \|U(hf)\|_2^2 \\
&\le \|U\|^2 \|hf\|_2^2 \\
&= \|U\|^2 \int_G |hf(x)|^2 dx \\
&= \|U\|^2 \int_{G/\varphi(L)} \sum_{k\in L} |hf(x\varphi(k^{-1}))|^2 d\dot{x} \\
&= \|U\|^2 \int_{G/\varphi(L)} |h(\dot{x})|^2 \|f\|_\varphi^2(\dot{x})d\dot{x},
\end{aligned}$$

which obviously completes the proof. □

Theorem 3.4. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(G)$. The following are equivalent.*

(1) *$(f_n)_{n\in\mathbb{N}} \subseteq L^2(G)$ is φ-complete, and there exist positive constants A and B such that for any sequence $\{h_n\}_{n\in\mathbb{N}}$ in $L^\infty(G/\varphi(L))$ with $\sum_{n=1}^{\infty}|h_n(\dot{x})|^2<\infty$ a.e. one has*

$$A\sum_{n=1}^{\infty}|h_n|^2 \leq \|\sum_{n=1}^{\infty}h_nf_n\|_\varphi^2 \leq B\sum_{n=1}^{\infty}|h_n|^2 \quad a.e. \tag{3.1}$$

(2) *$(f_n)_{n\in\mathbb{N}}$ is a φ-Riesz basis.*

(3) *$(M_\gamma f_n)_{\gamma\in\varphi(L)^\perp,n\in\mathbb{N}}$ is a Riesz basis in $L^2(G)$.*

Proof. (1)$\Rightarrow$ (2) Let $(e_n)_{n\in\mathbb{N}}$ be a φ-orthonormal basis in $L^2(G)$. Then by Theorem 1.2 (3), $\overline{span}^{\|\cdot\|_\varphi}(e_n) = L^2(G)$. Define $U : \overline{span}^{\|\cdot\|_\varphi}(e_n) \to L^2(G)$ by $U(\sum_{n=1}^{\infty}h_ne_n)=\sum_{n=1}^{\infty}h_nf_n$, where $\{h_n\}_{n\in\mathbb{N}}\subseteq L^\infty(G/\varphi(L))$ with $\sum_{n=1}^{\infty}|h_n(\dot{x})|^2<\infty$ a.e. Then U is bounded. In fact, by (3.1)

$$\begin{aligned}\|U(\textstyle\sum_{n=1}^{\infty}h_ne_n)\|_\varphi^2 &= \|\textstyle\sum_{n=1}^{\infty}h_nf_n\|_\varphi^2\\ &\leq B\textstyle\sum_{n=1}^{\infty}|h_n|^2\\ &= B\|\textstyle\sum_{n=1}^{\infty}h_ne_n\|_\varphi^2, \quad a.e.,\end{aligned}$$

and so

$$\begin{aligned}\|U(\textstyle\sum_{n=1}^{\infty}h_ne_n)\|_2^2 &= \int_{G/\varphi(L)}\|U(\textstyle\sum_{n=1}^{\infty}h_ne_n)\|_\varphi^2(\dot{x})d\dot{x}\\ &\leq B\int_{G/\varphi(L)}\|\textstyle\sum_{n=1}^{\infty}h_ne_n\|_\varphi^2(\dot{x})d\dot{x}\\ &= B\|\textstyle\sum_{n=1}^{\infty}h_ne_n\|_2^2,\end{aligned}$$

for any $\{h_n\}_{n\in\mathbb{N}}\subseteq L^\infty(G/\varphi(L))$ with $\sum_{n=1}^{\infty}|h_n(\dot{x})|^2<\infty$ a.e. That is, $\|U\|\leq\sqrt{B}$. Now define $S : L^2(G)(=\overline{span}^{\|\cdot\|_\varphi}(f_n)) \to L^2(G)$ by $S(\sum_{n=1}^{\infty}h_nf_n)=\sum_{n=1}^{\infty}h_ne_n$, where $\{h_n\}_{n\in\mathbb{N}}\subseteq L^\infty(G/\varphi(L))$ and $\sum_{n=1}^{\infty}|h_n(\dot{x})|^2<\infty$ a.e. Hence by (3.1) we get

$$\begin{aligned}\|S(\textstyle\sum_{n=1}^{\infty}h_nf_n)\|_\varphi^2 &= \|\textstyle\sum_{n=1}^{\infty}h_ne_n\|_\varphi^2\\ &= \textstyle\sum_{n=1}^{\infty}|h_n|^2\\ &\leq 1/A\|\textstyle\sum_{n=1}^{\infty}h_nf_n\|_\varphi^2, \quad a.e.\end{aligned}$$

This implies that S is bounded on $L^2(G)$ and $\|S\|\leq\sqrt{1/A}$. Also obviously, $SU=I$ and $US=I$ on $L^2(G)$. Hence U is a topological isomorphism, which is clearly φ-factorable and $U(e_n)=f_n$ for every $n\in\mathbb{N}$.

(2)$\Rightarrow$ (3) Choose a φ-orthonormal basis $(g_n)_{n\in\mathbb{N}}$ for $L^2(G)$ and the corresponding topological isomorphism U which is a φ-factorable operator and $U(g_n)=f_n$, for every $n\in\mathbb{N}$, as in the Definition 3.1. By Theorem 1.2 $(M_\gamma f_n)_{\gamma\in\varphi(L)^\perp,n\in\mathbb{N}}$ is an orthonormal basis for $L^2(G)$, and since U is φ-factorable $U(M_\gamma g_n)=M_\gamma U(g_n)=M_\gamma f_n$, for every $n\in\mathbb{N}$, $\gamma\in\varphi(L)^\perp$. So $(M_\gamma f_n)_{\gamma\in\varphi(L)^\perp,n\in\mathbb{N}}$ is a Riesz basis.

(3)$\Rightarrow$ (2) Let $S_{\varphi(L)}$ be a fundamental domain for $\varphi(L)$ (for the definition of a fundamental domain and a proof of existence see [14]). By [14, Theorem 3.1.7], the system $(M_\gamma T_{\varphi(k)}\chi_{S_{\varphi(L)}})_{k\in L,\gamma\in\varphi(L)^\perp}$ is an orthonormal basis for $L^2(G)$. Define $U : L^2(G)\to L^2(G)$ by $U(M_{\gamma_m}T_{\varphi(k_n)}\chi_{S_{\varphi(L)}})=M_{\gamma_m}f_n$, $m,n\in\mathbb{N}$. Using Lemma 2.2, U is a φ-factorable operator. Moreover, by Proposition 1.2, $(T_{\varphi(k)}\chi_{S_{\varphi(L)}})_{k\in L}$ is a φ-orthonormal basis for $L^2(G)$, and obviously $U(T_{\varphi(k_n)}\chi_{S_{\varphi(L)}})=f_n$, for every $n\in\mathbb{N}$. Finally since $(M_\gamma f_n)_{\gamma\in\varphi(L)^\perp,n\in\mathbb{N}}$ is a Riesz basis, U is a topological isomorphism.

(2)$\Rightarrow$ (1) Suppose $(e_n)_{n\in\mathbb{N}}$ is a φ-orthonormal basis and U is the corresponding topological isomorphism which is a φ-factorable operator and $U(e_n) = f_n$, for every $n \in \mathbb{N}$, as in the Definition 3.1. Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence in $L^\infty(G/\varphi(L))$ with $\sum_{n=1}^\infty |h_n(\dot{x})|^2 < \infty$ for a.e. $\dot{x} \in G/\varphi(L)$. Then using Lemma 3.3

$$\begin{aligned} \|\sum_{n=1}^\infty h_n f_n\|_\varphi^2 &= \|\sum_{n=1}^\infty h_n U(e_n)\|_\varphi^2 \\ &= \|U(\sum_{n=1}^\infty h_n e_n)\|_\varphi^2 \\ &\leq \|U\|^2 \|\sum_{n=1}^\infty h_n e_n\|_\varphi^2 \\ &= \|U\|^2 \sum_{n=1}^\infty |h_n|^2, \quad a.e. \end{aligned}$$

On the other hand

$$\begin{aligned} \sum_{n=1}^\infty |h_n|^2 &= \|\sum_{n=1}^\infty h_n e_n\|_\varphi^2 \\ &= \|U^{-1}U(\sum_{n=1}^\infty h_n e_n)\|_\varphi^2 \\ &\leq \|U^{-1}\|^2 \|U(\sum_{n=1}^\infty h_n e_n)\|_\varphi^2 \\ &= \|U^{-1}\|^2 \|\sum_{n=1}^\infty h_n f_n\|_\varphi^2, \quad a.e. \end{aligned}$$

So (3.1) holds. Moreover $(f_n)_{n\in\mathbb{N}}$ is φ-complete. Indeed, given any $f \in L^2(G)$, there exists a unique $g \in L^2(G)$ with $U(g) = f$ (since U is one-to-one and onto). Write $g = \sum_{n=1}^\infty [g, e_n]_\varphi e_n$ as in Theorem 1.2. Then $h_n = [g, e_n]_\varphi \in L^\infty(G/\varphi(L))$ for every $n \in \mathbb{N}$ and by Bessel's Inequality $\sum_{n=1}^\infty |h_n(\dot{x})|^2 \leq \|f\|_\varphi(\dot{x}) < \infty$ for a.e. $\dot{x} \in G/\varphi(L)$. Also $f = U(g) = U(\sum_{n=1}^\infty h_n e_n) = \sum_{n=1}^\infty h_n U(e_n) = \sum_{n=1}^\infty h_n f_n$, showing that $\overline{span}^{\|\cdot\|_\varphi}(f_n) = L^2(G)$. This completes the proof.

□

The next section is devoted to an application of the φ-bracket product to Weyl Heisenberg systems.

4. Applications to Weyl- Heisenberg Frames

In this section we investigate Weyl Heisenberg frames with regard to the φ-bracket product. For general references on Weyl Heisenberg frames on $\mathbb{R}$ we refer to the survey articles [1, 2].

Suppose L_1 and L_2 are two uniform lattices in G, $g \in L^2(G)$ and $T_{\varphi(k)}g$ is the translation of g by $\varphi(k)$. We call $(M_\gamma T_{\varphi(k)}g)_{\gamma\in\varphi(L_2)^\perp, k\in L_1}$, *a Weyl Heisenberg system (Gabor system).* If this system is a frame in $L^2(G)$ we call it a *Weyl Heisenberg frame.* In this case the frame operator associated with it is defined as $S(f) = \sum_{\gamma\in\varphi(L_2)^\perp} \sum_{k\in L_1} < f, M_\gamma T_{\varphi(k)}g > M_\gamma T_{\varphi(k)}g$. We would like to consider Weyl-Heisenberg frame Identity and the frame operator of a Weyl-Heisenberg frame in terms of the φ-bracket product. The following proposition is an extension of Weyl-Heisenberg frame Identity with regard to the φ-bracket product; see [5, 11, 4].

Proposition 4.1. *Let L_1 and L_2 be two uniform lattices in G. Let $g \in L^2(G)$ be φ-bounded. Then for every $f \in L^2(G)$ which is bounded and compactly supported we have:*

(4.1)
$$\sum_{k\in L_1} \sum_{\gamma\in\varphi(L_2)^\perp} |< f, M_\gamma T_{\varphi(k)}g >|^2 = \sum_{l\in L_2} \int_{G/\varphi(L_1)} [T_{\varphi(l^{-1})}f, f]_{\varphi,L_1}(\dot{x})[g, T_{\varphi(l^{-1})}g]_{\varphi,L_1}(\dot{x})d\dot{x},$$

where $[f, g]_{\varphi,L_i}(\dot{x}) = \sum_{k\in L_i} f\overline{g}(x\varphi(k^{-1}))$, $i = 1, 2$.

Proof. For $k \in L_1$, using The Plancherel Theorem we have
$\sum_{\gamma\in\varphi(L_2)^{\perp}} | < f, M_\gamma T_{\varphi(k)} g > |^2$
$= \sum_{\gamma\in\varphi(L_2)^{\perp}} | \int_G f(x)\overline{M_\gamma T_{\varphi(k)} g(x)} dx|^2$
$= \sum_{\gamma\in\varphi(L_2)^{\perp}} | \int_{G/\varphi(L_2)} \sum_{\varphi(l)\in\varphi(L_2)} f(x\varphi(l))\overline{g}(x\varphi(lk^{-1}))\overline{\gamma}(x) d\dot{x}|^2$
$= \sum_{\gamma\in\varphi(L_2)^{\perp}} |\hat{F}_k(\gamma)|^2$
$= ||\hat{F}_k||^2_{L^2(\widehat{G/\varphi(L_2)})}$
$= ||F_k||^2_{L^2(G/\varphi(L_2))}$,
where $F_k(x) = \sum_{\varphi(l)\in\varphi(L_2)} f(x\varphi(l))\overline{g}(x\varphi(lk^{-1}))$. So we get
$\sum_{k\in L_1}\sum_{\gamma\in\varphi(L_2)^{\perp}} | < f, M_\gamma T_{\varphi(k)} g > |^2$
$= \sum_{k\in L_1}\int_{G/\varphi(L_2)} |\sum_{\varphi(l)\in\varphi(L_2)} f(x\varphi(l))\overline{g}(x\varphi(lk^{-1}))|^2 d\dot{x}$
$= \sum_{k\in L_1}\int_{G/\varphi(L_2)} \sum_{\varphi(l)\in\varphi(L_2)} \overline{f}(x\varphi(l))g(x\varphi(lk^{-1}))\sum_{\varphi(m)\in\varphi(L_2)} f(x\varphi(m))\overline{g}(x\varphi(mk^{-1}))d\dot{x}$
$(put\ m = nl)$
$= \sum_{k\in L_1}\int_{G/\varphi(L_2)} \sum_{\varphi(l)\in\varphi(L_2)} \overline{f}(x\varphi(l))g(x\varphi(lk^{-1}))\sum_{\varphi(n)\in\varphi(L_2)} f(x\varphi(nl))\overline{g}(x\varphi(nlk^{-1}))d\dot{x}$
$= \sum_{k\in L_1}\int_G \bar{f}(x)g(x\varphi(k^{-1}))\sum_{\varphi(n)\in\varphi(L_2)} f(x\varphi(n))\overline{g}(x\varphi(nk^{-1}))dx$
$= \sum_{n\in L_2}\int_G \bar{f}(x)f(x\varphi(n))\sum_{k\in L_1} g(x\varphi(k^{-1}))\overline{g}(x\varphi(nk^{-1}))dx$
$= \sum_{n\in L_2}\int_G \bar{f}(x)f(x\varphi(n))[g, T_{\varphi(n^{-1})}g]_{\varphi,L_1}(x)dx$
$= \sum_{n\in L_2}\int_{G/\varphi(L_1)} \sum_{\varphi(l)\in\varphi(L_1)} \overline{f}(x\varphi(l))T_{\varphi(n^{-1})}f(x\varphi(l))[g, T_{\varphi(n^{-1})}g]_{\varphi,L_1}(\dot{x})d\dot{x}$
$= \sum_{n\in L_2}\int_{G/\varphi(L_1)} [T_{\varphi(n^{-1})}f, f]_{\varphi,L_1}(\dot{x})[g, T_{\varphi(n^{-1})}g]_{\varphi,L_1}(\dot{x})d\dot{x}.$

□

The ideas in the proof of Proposition 4.1 can be used to modify [11, Theorem 3.6], which leads to the following corollary; (see also [3]).

Corollary 4.2. *Let L_1 and L_2 be two uniform lattices in G. Let $g \in L^2(G)$ such that*

$$B := sup_{\dot{x}\in G/\varphi(L_1)} \sum_{k_2\in L_2} |[g, T_{\varphi(k_2)}g]_{\varphi,L_1}(\dot{x})| < \infty, \ and$$
$$A := inf_{\dot{x}\in G/\varphi(L_1)}[||g||^2_{\varphi,L_1}(\dot{x}) - \sum_{1_G\neq k_2\in L_2} |[g, T_{\varphi(k_2)}g]_{\varphi,L_1}(\dot{x})|] > 0. \tag{4.2}$$

Then $(M_\gamma T_{\varphi(k)} g)_{k\in L_1,\gamma\in\varphi(L_2)^{\perp}}$ is a Weyl-Heisenberg frame with bounds A, B.

It is useful to note also that the Weyl Heisenberg system has the following property.

Proposition 4.3. *Let L_1 and L_2 be two uniform lattices in G. If $f, g \in L^2(G)$ and g is φ-bounded then*

$$\sum_{\gamma\in\varphi(L_2)^{\perp}}\sum_{k\in L_1} | < f, M_\gamma T_{\varphi(k)} g > |^2 = \sum_{k\in L_1} ||[f, T_{\varphi(k)} g]_{\varphi,L_2}||^2_{L^2(G/\varphi(L_2))}. \tag{4.3}$$

Proof. Using The Plancherel Theorem we have the following calculations which proves (4.3).

$$\begin{aligned}
&\sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1}|<f,M_\gamma T_{\varphi(k)}g>|^2\\
&=\sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1}|\int_G f(x)\overline{T_{\varphi(k)}g}(x)\overline{\gamma}(x)dx|^2\\
&=\sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1}|\int_{G/\varphi(L_2)}\sum_{\varphi(l)\in\varphi(L_2)}f(x\varphi(l))\overline{T_{\varphi(k)}g}(x\varphi(l))\overline{\gamma}(x)d\dot{x}|^2\\
&=\sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1}|\int_{G/\varphi(L_2)}[f,T_{\varphi(k)}g]_{\varphi,L_2}(\dot{x})\overline{\gamma}(\dot{x})d\dot{x}|^2\\
&=\sum_{\gamma\in\varphi(L_2)^\perp}\sum_{k\in L_1}|\widehat{[f,T_{\varphi(k)}g]}_{\varphi,L_2}(\gamma)|^2\\
&=\sum_{k\in L_1}||\widehat{[f,T_{\varphi(k)}g]}_{\varphi,L_2}||^2_{L^2(\widehat{G/\varphi(L_2)})}\\
&=\sum_{k\in L_1}||[f,T_{\varphi(k)}g]_{\varphi,L_2}||^2_{L^2(G/\varphi(L_2))}.
\end{aligned}$$

□

In the sequel we will identify the frame operator of a Weyl-Heisenberg frame. For this we need a couple of lemmas.

Lemma 4.4. *Suppose $g\in L^2(G)$ is φ-bounded and φ-periodic. Let L be a uniform lattice in G. Then*

$$\sum_{\gamma\in\varphi(L)^\perp}<f,M_\gamma g>M_\gamma g=[f,g]_\varphi g \quad \text{a.e.} \quad \text{for all } f\in L^2(G), \tag{4.4}$$

where the series converges in $L^2(G)$. In particular, if $||g||_\varphi=1$ a.e. and P is the orthogonal projection onto $\overline{span}\{M_\gamma g\}_{\gamma\in\varphi(L)^\perp}$, then $Pf=[f,g]_\varphi g$ a.e.

Proof. Let $f\in L^2(G)$. By (1.3) we have $\sum_{\gamma\in\varphi(L)^\perp}<f,M_\gamma g>\gamma(\dot{x})=\sum_{\gamma\in\varphi(L)^\perp}\widehat{[f,g]_\varphi}(\gamma)\gamma(\dot{x})=[f,g]_\varphi(\dot{x})$, for a.e. $\dot{x}\in G/\varphi(L)$. Hence (4.4) holds, where the convergence of the series in $L^2(G)$ follows from Proposition 1.1. In particular, if $||g||_\varphi=1$ then $(M_\gamma g)_{\gamma\in\varphi(L)^\perp}$ is an orthonormal basis for $\overline{span}\{M_\gamma g\}_{\gamma\in\varphi(L)^\perp}$. So $Pf=\sum_{\gamma\in\varphi(L)^\perp}<f,M_\gamma g>M_\gamma g=[f,g]_\varphi g$ a.e. □

Lemma 4.5. *Let L_1 and L_2 be two uniform lattices in G, $g\in L^\infty(G/\varphi(L_1))$ and $(M_\gamma T_{\varphi(k)}g)_{\gamma\in\varphi(L_1)^\perp,k\in L_2}$ be a Bessel sequence with bound B in $L^2(G)$. Then $||g||^2_{\varphi,L_2}\le B$.*

Proof. Let $f\in L^2(G)$ be φ-periodic and $k\in L_2$. Then $f\cdot T_{\varphi(k)}\overline{g}\in L^2(G/\varphi(L_1))$. Since $\varphi(L_1)^\perp$ is an orthonormal basis for $L^2(G/\varphi(L_1))$ we have

$$\begin{aligned}
\sum_{\gamma\in\varphi(L_1)^\perp}|<f\cdot T_{\varphi(k)}\overline{g},M_\gamma>|^2 &= ||f\cdot T_{\varphi(k)}\overline{g}||^2_{L^2(G/\varphi(L_1))}\\
&= \int_{G/\varphi(L_1)}|f(x)|^2|g(x\varphi(k^{-1}))|^2d\dot{x}.
\end{aligned}$$

So

$$\begin{aligned}
\sum_{\gamma\in\varphi(L_1)^\perp,k\in L_2}|<f,M_\gamma T_{\varphi(k)}g>|^2 &= \sum_{\gamma\in\varphi(L_1)^\perp,k\in L_2}|<f\cdot T_{\varphi(k)}\overline{g},M_\gamma>|^2\\
&= \int_{G/\varphi(L_1)}|f(x)|^2\sum_{k\in L_2}|g(x\varphi(k^{-1})|^2d\dot{x}\\
&= \int_{G/\varphi(L_1)}|f(x)|^2||g||^2_{\varphi,L_2}(x)d\dot{x}.
\end{aligned} \tag{4.5}$$

On the other hand

$$\sum_{\gamma\in\varphi(L_1)^\perp,k\in L_2}|<f,M_\gamma T_{\varphi(k)}g>|^2\le B||f||^2_{L^2(G/\varphi(L_1))}. \tag{4.6}$$

Hence (4.5) and (4.6) imply that $||g||^2_{\varphi,L_2}\le B$, a.e. □

Whence the frame operator of a Weyl-Heisenberg frame is given by the following theorem.

Theorem 4.6. *Let L_1 and L_2 be two uniform lattices in G and $g \in L^\infty(G/\varphi(L_1))$. Suppose $(M_\gamma T_{\varphi(k)} g)_{\gamma \in \varphi(L_1), k \in L_2}$ is a Weyl-Heisenberg frame with the frame operator S. Then S has the form*

$$S(f) = \sum_{k \in L_2} [f, T_{\varphi(k)} g]_{\varphi, L_1} T_{\varphi(k)} g, \tag{4.7}$$

where the series converges unconditionally in $L^2(G)$.

Proof. By Lemma 4.5, $T_{\varphi(k)} g$ is φ-bounded, so we can use Lemma 4.4 to obtain

$$\begin{aligned} S(f) &= \textstyle\sum_{\gamma \in \varphi(L_1)^\perp, k \in L_2} < f, M_\gamma T_{\varphi(k)} g > M_\gamma T_{\varphi(k)} g \\ &= \textstyle\sum_{k \in L_2} [f, T_{\varphi(k)} g]_{\varphi, L_1} T_{\varphi(k)} g. \end{aligned}$$

□

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