The Capacity Region for Broadcast Channels with Conditionally Independent Message Sets

G.A.Hodtani

Department of Electrical Engineering, Sharif University of Technology, Tehran, Iran
Department of Electrical Engineering, Ferdowsi University of Mashhad, Mashhad, Iran
hodtani@yahoo.com

Abstract-*A more general class of broadcast channels with conditionally independent message sets is defined and its capacity region is determined using Marton's inner bound and the outer bound considered by Nair and El Gamal. The capacity region for this class of channels includes the capacity region for broadcast channels with degraded message sets, semi-deterministic broadcast channels, deterministic broadcast channels with and without common information, less noisy and (special) more capable broadcast channels.*

I. INTRODUCTION

The two receiver memoryless broadcast channel $(\mathcal{X}, p(y,z|x), \mathcal{Y}\times\mathcal{Z})$ consists of three finite sets $\mathcal{X}$ (input alphabet), $\mathcal{Y},\mathcal{Z}$ (output alphabets) and a collection of probability distributions $p(y,z|x)$ on $\mathcal{Y}\times\mathcal{Z}$, one for each $x\in\mathcal{X}$. Without loss of generality, it is assumed that the channel components are independent, i.e. , given an input letter x, two output letters y and z are generated independently of each other. The nth extension of the broadcast channel $(\mathcal{X}, p(y,z|x), \mathcal{Y}\times\mathcal{Z})$ is the broadcast channel $\left(\mathcal{X}^n, p\left(\underline{y},\underline{z}\middle|\underline{x}\right), \mathcal{Y}^n\times\mathcal{Z}^n\right)$, where $\underline{x}$, $\underline{y}$ and $\underline{z}$ indicate

the vectors $\underline{x}=(x_1,\cdots,x_n)$, $\underline{y}=(y_1,\cdots,y_n)$ and $\underline{z}$

$$=(z_1,\cdots,z_n) \text{ respectively and } p\left(\underline{y},\underline{z}\middle|\underline{x}\right)=\prod_{i=1}^{n}p(y_i,z_i|x_i) .$$

Three sources W,U,V (random variables) are defined on the set of integers $\mathcal{W}=[1,2^{nR_0}]$, $\mathcal{U}=[1,2^{nR_1}]$, $\mathcal{V}=[1,2^{nR_2}]$, respectively, where $u\in\mathcal{U}$ and $v\in\mathcal{V}$ are private messages and $w\in\mathcal{W}$ is a common message. We assume that W,U,V are generated independently and equiprobably over $\mathcal{W},\mathcal{U},\mathcal{V}$, respectively. Also for the random variables $(WUVXYZ)$, we have Markov chains

$$WUV\longrightarrow X\longrightarrow YZ \qquad (1)$$

Therefore, a broadcast channel $(\mathcal{X}, p(y,z|x), \mathcal{Y}\times\mathcal{Z})$ can be shown by the Fig.1 .

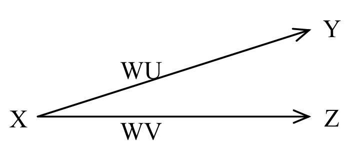

Fig.1 Broadcast channel

A $\left((2^{nR_0},2^{nR_1},2^{nR_2}),n\right)$ code for a broadcast channel consists of three sets of integers $\mathcal{W}=[1,2^{nR_0}]$, $\mathcal{U}=[1,2^{nR_1}]$, $\mathcal{V}=[1,2^{nR_2}]$, an encoding function $\underline{x}:\mathcal{W}\times\mathcal{U}\times\mathcal{V}\longrightarrow\mathcal{X}^n$

and two decoding functions

$$g_1:\mathcal{Y}^n\longrightarrow\mathcal{W}\times\mathcal{U}\ ;\ g_1(\underline{Y})=\left(\widehat{W},\widehat{U}\right) \qquad (2)$$

$$g_2:\mathcal{Z}^n\longrightarrow\mathcal{W}\times\mathcal{V}\ ;\ g_2(\underline{Z})=\left(\widehat{\widehat{W}},\widehat{V}\right). \qquad (3)$$

The set $\{\underline{x}(w,u,v) : (w,u,v)\in\mathcal{W}\times\mathcal{U}\times\mathcal{V}\}$ is called the set of codewords. Assuming a uniform distribution on the set of messages $\mathcal{W}\times\mathcal{U}\times\mathcal{V}$. we can define $P_{e_1}^n$ and $P_{e_2}^n$ to be the average probabilities of error at the decoders g_1 and g_2 respectively:

$$P_{e_1}^n=\frac{1}{2^{n(R_0+R_1+R_2)}}\sum_{(w,u,v)\in\mathcal{W}\times\mathcal{U}\times\mathcal{V}}P\left(g_1(\underline{Y})\neq(w,u)\middle|(w,u,v)\ sent\right) \quad (4)$$

$$P_{e_2}^n=\frac{1}{2^{n(R_0+R_1+R_2)}}\sum_{(w,u,v)\in\mathcal{W}\times\mathcal{U}\times\mathcal{V}}P\left(g_2(\underline{Z})\neq(w,v)\middle|(w,u,v)\ sent\right) \quad (5)$$

The rate (R_0,R_1,R_2) is said to be achievabe by a broadcast channel if, for any $\varepsilon>0$, there exists for all sufficiently large n, a $\left((2^{nR_0},2^{nR_1},2^{nR_2}),n\right)$ code such that $\max\left\{P_{e_1}^n,P_{e_2}^n\right\}<\varepsilon$.

The capacity region C for the broadcast channel is the set of all achievable rates (R_0,R_1,R_2) .

The broadcast channel was first introduced by Cover in his pioneering work [1]. Since then the problem of determining the capacity region has been investigated extensively and complete solutions for special classes, including the degraded broadcast channels [2],[3],[4], broadcast channels with degraded message sets [5], less noisy [6], more capable [7], deterministic with no common message [8],[9], semi-deterministic [10], deterministic with common massage [11] channels. Also, general inner bounds [12],[13],[14] and outer bounds [14],[15] have been established. Recently an outer bound [16] has been found that is strictly tighter than other outer bounds.

Additionally, broadcast channels with confidential messages [17], broadcast channels with cooperating decoders [18], wide-band broadcast channels [19], [20] and broadcast channels with relay [21], [22] have been studied.

In this paper, we define a more general class of broadcast channels and determine its capacity region using the Marton's inner bound [14] and Nair, El Gamal's outer bound [16]. Then, we show that this capacity region includes the capacity regions for the broadcast channels with degraded message sets, semi-deterministic, deterministic with and without common message, less noisy and (special) more capable broadcast channels. Finally, a conclusion is prepared.

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II. Broadcast Channels with Conditionally Independent Message Sets

First, we recall (a) the known capacity theorems for broadcast channels, (b) two theorems, the inner bound theorem [14, theorem 2] and the other, the outer bound theorem [16, theorems 2.1 and 3.1] and then (c) introduce the definition of broadcast channel with conditionally independent message sets and derive its capacity region and special cases.

(a) The Known Capacity Theorems for Broadcast Channels:

(a-1) The degraded broadcast channel [2],[3],[4] : The capacity region of the degraded broadcast channel $X \longrightarrow Z \longrightarrow Y$ (in Fig.1, $W = \emptyset$, $V = X$) is the convex hull of the closure of all (R_1 , R_2) satisfying :

$$R_1 \leq I(U;Y) \,, R_2 \leq I(X;Z|U)$$

for some joint distribution $p(u)\, p(x|u)\, p(y,z|x)$.

(a-2) Capability degraded broadcast channels (less noisy and more capable broadcast channels) :

(i) By $W = \emptyset$, $V = X$ in Fig.1, if the condition $I(U;Y) \leq I(U;Z)$ is satisfied for every $p(u)\, p(x|u)\; p(y,z|x)$, then, the channel is said to be a less noisy channel the capacity region of which is given by :

$R_1 \leq I(U;Y) \,, R_2 \leq I(X;Z|U)$ [6].

(ii) In Fig.1 ($U = \emptyset$, $WV = X$), if the condition $I(X;Y) \leq I(X;Z)$ is satisfied for every probability distribution on $\mathcal{X}$, then, the channel is said to be more capable channel the capacity region of which is given by :

$R_0 \leq I(W;Y) \,, R_0 + R_2 \leq I(X;Z) \,, R_0 + R_2 \leq I(W;Y) + I(X;Z|W)$ [7].

(a-3) Broadcast channel with degraded message sets: By $U = \emptyset$, Fig.1 indicates a broadcast channel with degraded message sets and $R_1 = 0$. By $WV = X$, its capacity region is : $R_0 \leq I(W;Y) \,, R_0 + R_2 \leq I(X;Z) \,, R_2 \leq I(X;Z|W)$, as in [5].

(a-4) Semi-deterministic broadcast channels [10] : In Fig.1, if $W = \emptyset$, $Y = f(X)$ or $Z = g(X)$, then, the channel is a semi-deterministic one and the capacity region for $Y = f(X)$ is :

$$R_y \leq H(Y) \,, R_z \leq I(V;Z) \,, R_y + R_z \leq I(V;Z) + H(Y|V)$$

(a-5) Deterministic broadcast channels with and without common message [8],[9],[11]: In Fig.1, if $Y = f(X)$ and $Z = g(X)$,then, the channel is a deterministic one by the capacity region:

$$R_y \leq I(W;Y) + H(Y|W) \,, R_z \leq I(W;Z) + H(Z|W) \,, R_y + R_z \leq min\,\{I(W;Y), I(W;Z)\} + H(YZ|W)$$

(b) The inner bound theorem [14, theorem 2] : The set of rate triples (R_0 , R_1 , R_2) satisfying :

$$\begin{cases} R_y = R_0 + R_1 \leq I(WU;Y) & \text{(6-a)} \\ R_z = R_0 + R_2 \leq I(WV;Z) & \text{(6-b)} \\ R_y + R_z = R_0 + R_1 + R_2 \leq min\,\{I(W;Y) , I(W;Z)\} + I(U;Y|W) + I(V;Z|W) - I(U;V|W) = A & \text{(6-c)} \end{cases}$$

for some $p(uvwx)\, p(y,z|x)$, is achievable.

The outer bound theorem [16, theorems 2.1 and 3.1] :
The set of rate triples (R_0 , R_1 , R_2) satisfying :

$$\begin{cases} R_0 \leq \min\{I(W;Y) , I(W;Z)\} & \\ R_0 + R_1 \leq I(UW;Y) & \text{(7-a)} \\ R_0 + R_2 \leq I(WV;Z) & \text{(7-b)} \\ R_0 + R_1 + R_2 \leq min\,\{\boldsymbol{B = I(UW;Y) + I(V;Z|UW) , C = I(VW;Z) + I(U;Y|VW)}\} = D & \text{(7-c)} \end{cases}$$

for some joint distribution of the form

$$\begin{cases} p(u,v,w,x)\, p(y,z|x) = p(u)\, p(v)\, p(w|v,u)\, p(x|u,v,w)\, p(y,z|x) \,, \mathcal{W} \neq \emptyset & \text{(8)} \\ p(u,v,x)\, p(y,z|x) = p(u,v)\, p(x|u,v)\, p(y,z|x) \,, \mathcal{W} = \emptyset & \text{(9)} \end{cases}$$

constitutes an outer bound to the capacity region for the broadcast channel.

(c) The broadcast channel with conditionally independent message sets and its capacity region:

Definition : A broadcast channel $(\mathcal{X} , p(y,z|x) , \mathcal{Y} \times \mathcal{Z})$ with auxiliary random variables W, U, V is said to be a broadcast channel with conditionally independent message sets if $(WUVXYZ)$ satisfy:

$$\begin{cases} V \longrightarrow WZ \longrightarrow U & \text{(10-a)} \\ I(W;Y) \leq I(W;Z) & \text{(10-b)} \end{cases}$$

for every general distribution of the form (8) or (9).

Theorem : The set of rate triples (R_0 , R_1 , R_2) satisfying:

$$\begin{cases} R_y = R_0 + R_1 \leq I(WU;Y) & \text{(11-a)} \\ R_z = R_0 + R_2 \leq I(WV;Z) & \text{(11-b)} \\ R_y + R_z = R_0 + R_1 + R_2 \leq I(WU;Y) + I(V;Z|UW) = B & \text{(11-c)} \end{cases}$$

constitutes the capacity region for a broadcast channel with conditionally independent message sets.

Remark : The relations (10-a,b) can be replaced by (12-a,b) below ($V \rightleftarrows U$ and $Y \rightleftarrows Z$) :

$$\begin{cases} U \longrightarrow WY \longrightarrow V & \text{(12-a)} \\ I(W;Z) \leq I(W;Y) & \text{(12-b)} \end{cases}$$

Then, from (11-a,b,c) by the replacements ($V \rightleftarrows U , Y \rightleftarrows Z , R_1 \rightleftarrows R_2$), the new capacity region (13-a,b,c) is obtained:

$$\begin{cases} R_y = R_0 + R_1 \leq I(WU;Y) & \text{(13-a)} \\ R_z = R_0 + R_2 \leq I(WV;Z) & \text{(13-b)} \\ R_y + R_z = R_0 + R_1 + R_2 \leq C = I(VW;Z) + I(U;Y|VW) & \text{(13-c)} \end{cases}$$

<u>The proof of theorem</u> :

Achievability : The relations (11-a,b) are trivial from (6-a,b). And (11-c) is obtained from (6-c) :

$$R_0 + R_1 + R_2 \leq A = min\,\{I(W;Y) , I(W;Z)\} + I(U;Y|W) + I(V;Z|W) - I(U;V|W) \overset{\text{(10-b)}}{=} I(UW;Y) + I(V;Z|W) - I(U;V|W) =$$

$$I(UW;Y) - H(V|WZ) + H(V|UW) \overset{\text{(10-a)}}{=} I(UW;Y) - H(V|WZU) + H(V|UW) = I(UW;Y) + I(V;Z|UW) = B,$$

therefore,

$$R_0 + R_1 + R_2 \leq A = B = I(UW;Y) + I(V;Z|UW) \quad (14)$$

Converse : The relations (11-a,b) are trivial from (7-a,b). And from (7-c) we have :

$$R_0 + R_1 + R_2 \leq D = min\,\{B, C\} \overset{(14)}{=} min\,\{A, C\} \leq A = B \,,$$

that is, the inner bound (6-c) coincides with the outer bound (7-c). Then the proof is complete.

Remark : For a broadcast channel with (12-a,b), the capacity region is (13-a,b,c), instead of (11-a,b,c).

Special cases of theorem :

(a) by $U = \emptyset$, Fig.1 indicates a broadcast channel with degraded message sets and $R_1 = 0$. By $WV = X$, we have :

$$\text{(11-a)} \implies R_0 \leq I(W;Y) \quad \text{(15-a)}$$

$$\text{(11-b)} \implies R_0 + R_2 \leq I(WV;Z) = I(X;Z) \quad \text{(15-b)}$$

(11-c) $\Rightarrow$ $R_0 + R_2 \leq I(W;Y) + I(V;Z|W) \leq I(W;Z) + I(V;Z|W) = I(WV;Z) = I(X;Z)$ (15-c)

(15-a,c) $\Rightarrow$ $R_2 \leq I(V;Z|W) = I(X;Z|W)$ (15-d)

That is, (15-a,b,d) show the capacity region for the broadcast channel with degraded message sets as in [5].

(b) If $Z = g(X)$, by $W = \emptyset$ and $V = Z$, (10-a,b) are satisfied and (11-a,b,c) give the capacity region for semi-deterministic broadcast channel as in [10] :

(11-a) $\Rightarrow$ $R_y \leq I(U;Y)$

(11-b) $\Rightarrow$ $R_z \leq H(Z)$

(11-c) $\Rightarrow$ $R_y + R_z \leq I(U;Y) + H(Z|U)$

- If $Y = f(X)$, by $Y = U$, $W = \emptyset$, the conditions (12-a,b) are satisfied and the corresponding capacity region (13-a,b,c) gives the capacity for semi-deterministic broadcast channel :

$$R_y \leq H(Y) , R_z \leq I(V;Z) , R_y + R_z \leq I(V;Z) + H(Y|V)$$

(c) If $Y = f(X)$ and $Z = g(X)$, by $W = \emptyset$, $V = Z$, $U = Y$, (11-a,b,c) give the capacity for deterministic broadcast channel without common message :

$$R_y \leq H(Y) , R_z \leq H(Z) , R_y + R_z \leq H(Y;Z)$$

(d) If $Y = f(X)$ and $Z = g(X)$, by $W \neq \emptyset$, $V = Z$, $Y = U$, (11-a,b,c) and (13-a,b,c) give the capacity region for deterministic broadcast channel with common message as in [11] :

$$\begin{aligned} R_y \leq I(W;Y) + H(Y|W) , R_z \\ \leq I(W;Z) + H(Z|W) , R_y + R_z \\ \leq min\,\{I(W;Y), I(W;Z)\} + H(YZ|W) \end{aligned}$$

(e) By $W = \emptyset$, $V = X$, (10-a) $\Rightarrow$ $V = X \longrightarrow Z \longrightarrow U$ (16),

and (16) and ((1) $\longrightarrow (YZ \longrightarrow X \longrightarrow U)$) $\Rightarrow$ $I(U;X) = I(U;Z)$ (17),

and ((1) $\longrightarrow (Y \longrightarrow X \longrightarrow U)$) $\Rightarrow$ $I(U;Y) \leq I(U;X)$ (18),

and (17, 18) $\Rightarrow$ $I(U;Y) \leq I(U;Z)$ $\Rightarrow$ channel is a less noisy channel [6],

and then

(11-a) $\Rightarrow$ $R_1 \leq I(U;Y)$

(11-b) $\Rightarrow$ $R_2 \leq I(V;Z) = I(X;Z)$

(11-c) $\Rightarrow$ $R_1 + R_2 \leq I(U;Y) + I(X;Z|U) \leq I(U;Z) + I(X;Z|U) = I(X;Z)$

So, we have: $R_1 \leq I(U;Y) , R_2 \leq I(X;Z|U)$.

(f)

(i) If (Z is more capable than Y) means that every private message to Y is always a common message for both Y and Z, then, we can put $U = \emptyset$ in Fig.1 and hence $WV = X$, then from (11-a,b,c), the capacity region as for more capable channels in [7] is as follows:

$$\begin{aligned} R_0 \leq I(W;Y) , R_0 + R_2 \leq I(X;Z) , R_0 + R_2 \\ \leq I(W;Y) + I(V;Z|W) \\ = I(W;Y) + I(X;Z|W) \end{aligned}$$

(ii) If merely we have (10-a,b), then, by $U = \emptyset$, $WV = X$ and special condition $V \longrightarrow W \longrightarrow Y$, we have :

$I(X;Z) = I(W;Z) + I(V;Z|W) \geq I(W;Y) + I(V;Y|W) = I(X;Y)$, that is, the channel is more capable.

III.CONCLUSION

A more general class of broadcast channels was defined the capacity region of which includes the capacity region for broadcast channels with degraded message sets, semi-deterministic, deterministic with and without common message, less noisy and (special) more capable broadcast channels.

REFERENCES

[1] T.M.Cover, "Broadcast Channels," IEEE Trans. Inform. Theory, vol.18, no.1, 1972.

[2] P.P.Bergmans, "Random Coding Theorem for Broadcast Channels with Degraded Components," IEEE Trans. Inform. Theory, vol.19, no.2, 1973.

[3] R.G.Gallager, "Capacity and Coding for Degraded Broadcast Channels," Probl. Peredac. Inform., 10(3):3-14, 1974.

[4] R.F.Ahlswede and J.Körner, "Source Coding with Side Information and a Converse for Degraded Broadcast Channels," IEEE Trans. Inform. Theory, vol.21, no.6, 1975.

[5] J.Körner and K.Marton, "General Broadcast Channels with Degraded Message Sets," IEEE Trans. Inform. Theory, vol.23, no.1, 1977.

[6] J.Körner and K.Marton, "A Source Network Problem Involving the Comparison of two Channels ii," Trans. Colloquim Inform. Theory, Keszthely, Hungary, Auguts, 1975.

[7] A.El Gamal, "The Capacity of a Class of Broadcast Channels," IEEE Trans. Inform. Theory, vol.25, no.2, March, 1979.

[8] K.Marton, "The Capacity Region of Deterministic Broadcast Channels," ISIT, 1977.

[9] M.S.Pinsker, "Capacity of Noiseless Broadcast Channels," Probl. Pered. Inform., 14(2):28-334, Apr.-Jun., 1978.

[10] S.I.Gelfand and M.S.Pinsker, "Capacity of a Broadcast Channel with one Deterministic Component," Probl. Inform. Trans., 16(1):17-25, Jan.-Mar., 1980.

[11] T.S.Han, "The Capacity Region for the Deterministic Broadcast Channel with a Common Message," IEEE Trans. Inform. Theory, vol.27, no.1, 1981.

[12] T.M.Cover, "An Acheivable Rate Region for the Broadcast Channel," IEEE Trans. Inform. Theory, vol.21, no.4, July, 1975.

[13] E.van.der.Meulen, "Random Coding Theorems for the Discrete Memoryless Broadcast Channel," IEEE Trans. Inform. Theory, vol.21, no.2, 1975.

[14] K.Marton, "A Coding Theorem for the Discrete Memoryless Broadcast Channel," IEEE Trans. Inform. Theory, vol.25, no.3, May, 1979.

[15] H.Sato, “An Outer Bound to the Capacity Region of Broadcast Channels,” IEEE Trans. Inform. Theory, IT-24:374-377, May, 1978.

[16] C.Nair and A.El Gamal, “An Outer Bound to the Capacity Region of the Broadcast Channel,” ISIT 2006.

[17] I.Csiszar and J.Körner, “Broadcast Channels with Confidential Messages,” IEEE Trans. Inform. Theory, vol.24, no.3, 1978.

[18] R.Dabora and S.D.Servetto, “Broadcast Channels with Cooperating Decoders,” Submitted to IEEE 2006.

[19] A.Sokolovsky and S.I.Bross, “Attainable Error Expanents for the Poisson Broadcast Channel with Degraded Message Sets,” ISIT 2002.

[20] A.Lapidoth and E.Telatar, “On Wide-Band Broadcast Channels,” IEEE Trans. Inform. Theory, vol.49, no.12, 2003.

[21] G.Kramer, M.Gastpar and P.Gupta, “Cooperative Strategies and Capacity Theorems for Relay Networks,” IEEE Trans. Inform. Theory, vol.51, no.9, 2005.

[22] Y.Liang and G.Kramer, “Rate Regions for Relay Broadcast Channels,” IEEE Trans. Inform. Theory, vol.53,no.10,2007.