A New Achievable Rate and A Certain Capacity Result for A Stochastic Two Relay Network With No Interference

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Abstract-Gastpar and Vetterli named the ad-hoc networks with one randomly selected source-destination pair as relay networks and they suggested using arbitrary network coding in order to increase the overall efficiency Here, the relay network is investigated with two relays and no interference and a new achievable rate is obtained using a new network coding (decode-and-broadcast). The obtained achievable rate (i) gives the lower bound for general relay channel, obtained by Cover and El Gamal, but with slight difference due to no interference assumption at the receiver (ii) includes the one relay rates of a two-level relay channel studied by Gupta and Kumar using point to point coding, (iii) includes the rates of two relay Aref network and other special two relay networks, (iv) meets the max-flow min-cut upper bound under certain additional assumptions resulting in certain capacity theorems which include the related previous capacities, (v) is validated by its consistency with previous results relevant to special cases of broadcast channels.

Key Words: stochastic (deterministic) two relay network, network coding

I. INTRODUCTION

Recently there has been much interest in studying relay networks and wireless relay communication scenarios [1], [2].Wireless networks with base stations (cell networks) are relatively well understood by considering the multiple access and the broadcast channels. However, the networks without base stations (ad-hoc networks) in which any node can act both as a terminal and as a relay for other transmissions are less well understood and have many open problems. Ad-hoc networks was studied in [3],[4] as networks having n nodes communicating with each other and multiple source-destination pairs to obtain an achievable rate region using point to point coding. In [5] mobile ad-hoc networks have been studied. In [6], ad-hoc networks with one randomly selected source-destination pair was named as relay networks. Also, in [6] arbitrary network coding was suggested to increase the overall efficiency and was used to obtain achievable rates and to derive upper bounds from max-flow min-cut theorem. Various problems of wireless ad-hoc networks have been studied in [32]-[38].

Here we consider the relay network introduced in [6] with two relays and no interference. This two relay network was studied in [7] with deterministic links and in [29] with one random link. Now we study the two relay network with randomness for all links to obtain a new achievable rate using a new network coding (decode-and-broadcast).In other words, we investigate the two-level relay channel studied in [4,Fig.1] using a new network coding instead of point to point coding.The obtained achievable rate gives the lower bound for the general relay channel obtained by Cover and El Gamal with slight difference due to no interference assumption at the receiver , includes the one relay rates of a two-level relay channel studied by Gupta and Kumar using point to point coding, includes the rates of two relay Aref network and other special two relay networks, meets the max-flow min-cut upper bound under certain additional assumptions resulting in certain capacity theorem which includes the related previous capacities and is validated by its consistency with previous results relevant to special cases of broadcast channels.

At first, in this section we recall some definitions, define the network model considered, describe the network coding, remember previous studies of the model and then we list our new results.

A. Some definitions

Relay network [6] : A relay network is a wireless network having one source-destination pair and some relays where the relays act as terminals (transmitters and receivers or cooperative nodes).

Stochastic (deterministic) relay network with no interference [7],[8],[29] : Stochastic (deterministic) relay network with no interference is a network where the output of every link is some stochastic (deterministic) function of only the input of that link.

So, a stochastic (deterministic) relay network with no interference is a special case of relay network introduced in [6]. Deterministic relay networks introduced in [7] have been named as Aref networks in [8].

Remarks: 1. The assumption of no interference is applicable only for broadcast channels and does not apply to multiple access channels. For this reason, we will use decode-and-broadcast coding (namely, the source and the relays act as broadcast channels).

2. In [9],[10], the relay networks have been studied with a more general definition of determinism (approximately Gaussian determinism).
3. In deterministic wired network with independent channels [11], we don't have broadcasting while in a stochastic (deterministic) relay network with no interference and in other wireless networks such as wireless erasure networks [12] we have broadcasting.

B. **Network model :**

Here, as mentioned in the introduction, we consider a stochastic two relay network with no interference (Fig.1) consisting of one source (x_0), one destination (y_3), two relays (x_1 and x_2) and seven stochastic links: the link between the source and the destination (y_{03}), two links between the relays (y_{12} , y_{21}), two links between the source and the relays (y_{01} , y_{02}) and two links between the relays and the destination (y_{13} , y_{23}). Actually, we treat a special case of general relay network introduced in [6].

The above two relay network consists of a source input alphabet $\mathcal{X}_0$, relay input alphabets $\mathcal{X}_1$, $\mathcal{X}_2$ and output alphabets $\mathcal{Y}_{01}$, $\mathcal{Y}_{02}$, $\mathcal{Y}_{03}$, $\mathcal{Y}_{12}$, $\mathcal{Y}_{21}$, $\mathcal{Y}_{13}$, $\mathcal{Y}_{23}$. The network is characterized by the probability distribution $p(y_{01} , y_{02} , y_{03} , y_{12} , y_{21} , y_{13} , y_{23} | x_0\, x_1\, x_2) = p(\cdot \mid \cdot)$, where x_0 , x_1 and x_2 indicate the source input, the first relay input and the second relay input, respectively and y_{01} , y_{02} , y_{03} , y_{12} , y_{21} , y_{13} , y_{23} indicate the outputs.

We assume that the network is memoryless, i.e, the present outputs depend on the messages, the previous inputs, and the previous outputs only through the present inputs.

We use $\underline{x}$ and $\underline{y}$ to represent the input vectors $(x_1 , \cdots , x_n)$ and the output vectors$(y_1 , \cdots , y_n)$, respectively, such that in Fig.1 $\underline{y}_1 = \left(\underline{y}_{01}\underline{y}_{21}\right)$, $\underline{y}_2 = \left(\underline{y}_{02}\underline{y}_{12}\right)$, $\underline{y}_3 = \left(\underline{y}_{03}\underline{y}_{13}\underline{y}_{23}\right)$, $\underline{x}_0 = (x_{01} , \cdots , x_{0n})$, $\underline{x}_1 = (x_{11} , \cdots , x_{1n})$, $\underline{x}_2 = (x_{21} , \cdots , x_{2n})$ and as in general multi-terminal networks [23, ch.14.10] $x_{0i} = f_{0i}(w)$, $x_{1i} = f_{1i}\left(y_1^{i-1}\right) = f_{1i}(y_{11} , \cdots , y_{1i-1})$ and $x_{2i} = f_{2i}\left(y_2^{i-1}\right) = f_{2i}(y_{22} , \cdots , y_{2i-1})$, $i = 1, \cdots , n$.

The message w with the rate R is the new message to be sent in each transmission block. The destination computes its message estimate $\hat{w}$ as a function of $\underline{y_3}$. Suppose that w has B_w bits. The capacity C is the supremum of rates $R = \frac{B_w}{n}$ at which the destination's message estimate $\hat{w}$ can be made to satisfy $Pr(\hat{w} \neq w) < \varepsilon$ for any positive ε . An upper bound for capacity is the max-flow min-cut upper bound [23, theorem 14.10.1].

Fig.2 illustrates the network model where the message w is sent and ultimately $\hat{w}$ is estimated at the receiver.

C. The network coding (decode-and-broadcast) :

We apply a new technique (decode-and-broadcast) established from the combination of superposition encoding [14]-[17] and random binning [19]-[22]. This technique is a generalization of the coding scheme used in [17, theorem 7, special case by $\hat{Y}_1 = \emptyset$, $V = X_2 \longrightarrow X_1$] to two relays, but with the addition of the cooperation between the relays (or adding the links y_{12} , y_{21}) . Therefore, at first we split the message w (as in [17], theorem 7) into $w = (w_{01} , w_{02} , w_{03})$ and secondly we allow the relays to broadcast for cooperating with each other and the receiver.

Suppose that $B-2$ block messages $w_1 , w_2 , \cdots , w_i , \cdots , w_{B-2}$, each block i of nR bits and the message $w_i = (w_{01i} , w_{02i} , w_{03i})$, are transmitted from the source x_0 to the destination y_3 during nB transmissions. Therefore, the total effective **single rate** R from the source x_0 to the destination y_3 in Fig.1 equals $\frac{R(B-2)}{B} \xrightarrow{B\to\infty} R = R_{01} + R_{02} + R_{03}$. Also, the same codebook is used in each block of transmission.

The assumption of no interference has been applied in the literature for reliable communication channels between nodes in models of ad-hoc wireless networks based on the geometric disc abstraction .For example, in [37] the outer bounds for the capacity have been determined with this assumption. Here we apply this assumption to the broadcast channels in the network (Fig.1). So, we use decode and broadcast coding, i.e., the relays (x_1 and x_2) act as transmitters for the broadcast channels and they broadcast what they decode from the source and the messages to each other, to the receiver and each other through x_1 and x_2 . In other words, two relays while cooperating with each other, cooperate with the transmitter to increase the total rate of transmission. The source (x_0) also acts as a transmitter for the receivers y_{01} and y_{02} and meanwhile it superimposes its message w_{03} intended only to the destination y_3 on its messages w_{01} , w_{02} intended for $y_{01}y_{02}$. The messages w_{01}, w_{02} are decoded by the first relay and the second relay respectively and then by the receiver, but the message w_{03} is decoded only by the receiver (y_3) .

For broadcasting source and relays, it is necessary to use auxiliary random variables $u_1 , v_1 , u_2 , v_2 , u_0$ and v_0 in relation with input random variables x_0 , x_1 and x_2 and to choose an appropriate joint distribution (distribution (A) in main theorem, section II).

D. The previous studies of the network model in Fig.1 :

The network in Fig.1 has been studied in :

D_1 . [7] when all of the links are deterministic and its capacity has been determined using deterministic broadcast channels [26], [27] and max-flow min-cut upper bound [23],
D_2 . [4] as two level relay channel by point to point coding without the link y_{21} in Fig.1,
D_3 . [6] **wi**th additive white Gaussian regime,
D_4 . [13] in a diamond topology, without the links y_{12} , y_{21} , y_{03} .

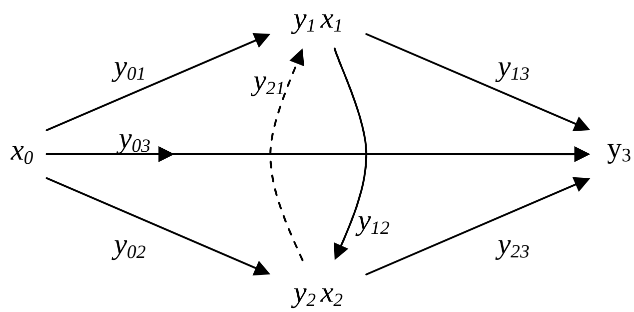


Fig.1 Two Relay Network

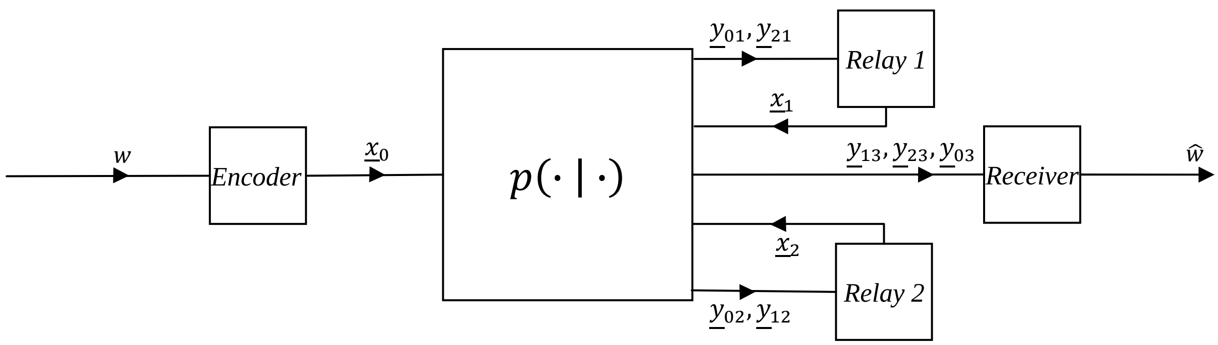


Fig.2 Network Model

E. New results :

The new achievable rate we have obtained (section II, main theorem) for the two relay network in Fig.1:
E_1 . Gives a lower bound for the general relay channel the same as obtained by Cover and El Gamal [17,theorm7,special case] , but with slight difference due to no interference assumption at the receiver and includes the one relay rates of a two-level relay channel studied by Gupta and Kumar using point to point coding,
E_2 . Includes the rate of deterministic two relay network [7] and other special two relay networks,
E_3 . Meets max-flow min-cut upper bound [23] under additional assumptions and
E_4 . Is validated by its consistency with previous results related to special cases of broadcast channels [24],[25].

The remainder of the paper is organized as follows. In section II, the main theorem is stated. In section III, the main theorem's corollaries are derived and explained. In section IV, we prove the theorems and section V prepares a conclusion.

II. MAIN THEOREM

Consider again the stochastic two relay network with no interference (Fig.1) where the source (x_0) and the relays (x_1 and x_2) act as broadcast channels and all of the links are stochastic.

Regarding the network (Fig.1), for every cut-set bound achieving capacity theorem, the converse can be proved from general upper-bound at least under special cases and only the achievability proof is of importance.

Therefore, by using binning method [19]-[22], superposition encoding [14]-[17] and combining all of the known coding techniques, we establish the decode-and-broadcast strategy and in order to apply this strategy we choose the appropriate distribution (the distribution (A) in the main theorem). By noting to the fact that the new message to be sent to the receiver in block i is $w_i = (w_{01i}, w_{02i}, w_{03i})$ with the rate $R = R_{01} + R_{02} + R_{03}$, we first introduce an achievable rate (main theorem) for R and then derive its special cases, e.g., some certain capacity theorems in section III.

Main Theorem : For the stochastic network (Fig.1) with no interference, the following rate is achievable:

$$\boldsymbol{R^* = \sup\ min\ \{R^*_{c_1}, R^*_{c_2}, R^*_{c_3}, R^*_{c_4}\}}$$

where $R^*_{c_1} = I(U_0; Y_{01}|X_1) + I(V_2; Y_{23}) + I(U_2; Y_{21}) - I(U_2; V_2) + H(Y_{03}Y_{13}Y_{23}|X_1X_2) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2),$

$R^*_{c_2} = I(V_0; Y_{02}|X_2) + I(V_1; Y_{13}) + I(U_1; Y_{12}) - I(U_1; V_1) + H(Y_{03}Y_{13}Y_{23}|X_1X_2) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2),$

$R_{c_3}^* = I(V_1; Y_{13}) + I(V_2; Y_{23}) + H(Y_{03}Y_{13}Y_{23}|X_1X_2) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2)$,
$R_{c_4}^* = H(Y_{03}Y_{13}Y_{23}V_0U_0|X_1X_2) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2) - H(U_0|X_1Y_{01}) - H(V_0|X_2Y_{02})$,
and supremum is taken over the joint distribution of the form :

$$p(u_1v_1u_2v_2u_0v_0x_0x_1x_2, y_{03}y_{13}y_{23}y_{12}y_{21}y_{01}y_{02}) = \underbrace{p(u_1v_1u_2v_2)\, p(x_1|u_1v_1)\, p(x_2|u_2v_2)p(u_0|x_1)p(v_0|x_2)p(x_0|u_0v_0x_1x_2)}_{A-1}\underbrace{p(y_{03}y_{13}y_{23}y_{12}y_{21}y_{01}y_{02}|x_0x_1x_2)}_{A-2}, \quad \text{(A)}$$

A-1 and A-2 represent input and random output distributions, respectively. A-1 satisfies :

$$X_2 \longrightarrow X_1 \longrightarrow U_0\ ,\ U_0X_1 \longrightarrow X_2 \longrightarrow V_0\ ,\ X_1U_1V_1 \longrightarrow U_2V_2 \longrightarrow X_2 \qquad \text{(B)}$$

Proof: See section IV.

Cut-set interpretation of the rates $R_{c_i}^*$, $i = 1, 2, 3, 4$ in R^*

$R_{c_1}^*, R_{c_2}^*, R_{c_3}^*, R_{c_4}^*$ are the achievable rates through the cut-sets $(y_{23}, y_{21}, y_{03}, y_{01})$, $(y_{13}, y_{12}, y_{03}, y_{02})$, (y_{13}, y_{23}, y_{03}), (y_{01}, y_{02}, y_{03}) in Fig.1, respectively.

III. MAIN THEOREM'S COROLLARIES

Now, we give some of the main theorem's corollaries (three corollaries).

Corollary 1: A lower bound for general relay channel and about the two-level relay channel

A lower bound for general relay channel

By $Y_{02} = X_2 = Y_{12} = Y_{21} = Y_{23} = \emptyset$, Fig.1 reduces to a relay channel [17] and from $R^*(V_0 = \emptyset, U_1 = U_2 = V_2 = \emptyset, V_1 = X_1)$, we obtain the following lower bound:

$$R_1 = \sup_{p(x_0x_1u_0)} \min\left\{\underbrace{I(U_0; Y_{01}|X_1) + I(X_0; Y_{03}Y_{13}|X_1U_0)}_{(1)}, \underbrace{I(X_1; Y_{13}) + I(X_0; Y_{03}Y_{13}|X_1)}_{(2)}\right\},$$

where (1) is the minimum of the first and the fourth terms in R^* ($U_0 \longrightarrow X_0X_1X_2 \longrightarrow Y_{03}Y_{13}Y_{23}$ in $A-2$) and (2) is obtained from the second or the third terms in R^*.

R_1 is slightly different, due to no interference assumption at the receiver, from the lower bound below [17, theorem 7, $\hat{Y}_1 = \emptyset, V = X_2 \longrightarrow X_1, U \longrightarrow U_0X_1, Y_1 \longrightarrow Y_{01}, Y = (Y_{03}Y_{13})$]:

$$R_2 = \sup_{p(x_0x_1u_0)} \min\left\{\underbrace{I(U_0; Y_{01}|X_1) + I(X_0; Y_{03}Y_{13}|X_1U_0)}_{(3)}, \underbrace{I(X_0X_1; Y_{03}Y_{13})}_{(4)}\right\},$$

where $(1) = (3)$ and $(2) < (4)$.

Remarks :

4-A special case of our main theorem ($Y_{12} = Y_{21} = \emptyset$) is a generalization of R_1 to two relays. In other words, we have generalized the lower bound R_1 to two relays, but with the addition of the cooperation between the relays by the links y_{12}, y_{21}.

5-R_1 gives the capacity –defining rates of the relay channels with known capacity (degraded: $U_0 = X_0$, reversely degraded: $U_0 = \emptyset$, full feedback: $U_0 = X_0$ and $Y_{03}Y_{13} \longrightarrow Y_{03}Y_{13}Y_{01}$, semi-deterministic: $U_0 = Y_{01}$ and orthogonal relay channels[28]: $U_0 = X_R$, $X_0 = (X_RX_D)$), but with no interference at the receiver. Therefore, our main theorem is a generalization of these channels to two relays, but with additional links y_{12}, y_{21} and no interference condition at the receiver.

6-$R_1 = R_2$ when the no interference condition ($X_1 \longrightarrow Y_{13} \longrightarrow Y_{03}$) is applied.

About the two-level relay channel [4] :

Point to point coding used in [4] for the two-level relay channel gives the rates of degraded and reversely degraded relay channels for a relay channel, but our new coding strategy gives a general lower bound for it.

Corollary 2: The rate of two relay Aref network and the rate regions for special broadcast channels :

We can validate the main theorem by its consistency with previous results in [7],[19],[25],[30],[31]. Here we choose the relevancy to two relay Aref network[7] and special broadcast channels [19],[25].

First, we assume that all of the links but y_{03} in Fig.1 are deterministic and determine the following achievable rate R_1^*(theorem 1). Then we derive from R_1^* the rate of two relay Aref network and explain its consistency with known results.

Theorem 1: For the network (Fig.1) with no interference, random y_{03} and assuming all other links to be deterministic, the following rate is achievable:

$$R_1^* = \sup\ min\left\{\begin{array}{l} H(Y_{23}\,Y_{21}) + H(Y_{01}|X_1) + H(Y_{03}|X_1X_2) - H(Y_{03}|X_0X_1X_2), \\ H(Y_{13}\,Y_{12}) + H(Y_{02}|X_2) + H(Y_{03}|X_1X_2) - H(Y_{03}|X_0X_1X_2), \\ H(Y_{13}) + H(Y_{23}) + H(Y_{03}|X_1X_2) - H(Y_{03}|X_0X_1X_2), \\ H(Y_{01}Y_{02}Y_{03}|X_1X_2) - H(Y_{03}|X_0X_1X_2) \end{array}\right\}$$

Where $y_{01} = f_{01}(x_0\, x_1), y_{02} = f_{02}(x_0x_2), y_{12} = f_{12}(x_1), y_{21} = f_{21}(x_2), y_{13} = f_{13}(x_1), y_{23} = f_{23}(x_2)$,

and supremum is taken over the joint distribution of the form : $p(x_0x_1x_2y_{01}y_{02}y_{03}y_{12}y_{21}y_{13}y_{23}) =$
$\underbrace{p(y_{12}y_{13}y_{23}y_{21})\ p(x_1|y_{12}y_{13})\ p(x_2|y_{23}y_{21})\ p(y_{01}|x_1)p(y_{02}|x_2)\ p(x_0|y_{01}y_{02}x_1x_2)}_{A_1-1}\ \underbrace{p(y_{03}|x_0x_1x_2)}_{A_1-2}$ (A₁)

A_1-1 and A_1-2 represent equivalent input and random output distributions, respectively. A_1-1 satisfies :
$X_2 \longrightarrow X_1 \longrightarrow Y_{01}\ ,\ Y_{01}X_1 \longrightarrow X_2 \longrightarrow Y_{02}\ ,\ X_1Y_{12}Y_{13} \longrightarrow Y_{21}Y_{23} \longrightarrow X_2$ (B_1),
and the distribution (A_1) has been obtained from the distribution (A) in the main theorem by $u_1 = y_{12}\,, v_1 = y_{13}\,, u_2 = y_{21}\,, v_2 = y_{23}\,, u_0 = y_{01}\,, v_0 = y_{02}$ and $y_{01} = f_{01}(x_0\,x_1), y_{02} = f_{02}(x_0x_2)$.
Proof: See section IV.

Special cases of theorem 1:

(a) ***The rate of two relay Aref network [7] :***

If we consider the network in Fig.1 with $y_{01} = f_{01}(x_0\,x_1), y_{02} = f_{02}(x_0x_2), y_{12} = f_{12}(x_1), y_{21} = f_{21}(x_2), y_{13} = f_{13}(x_1), y_{23} = f_{23}(x_2)$ and $\boldsymbol{y_{03} = f_{03}(x_0)}$,we have the distribution:
$p(x_0x_1x_2y_{01}y_{02}) = (A_1) = p(x_1x_2)\ p(x_0y_{01}y_{02}|x_1x_2)$ (C),
that is, (B_1) conditions hold and the inputs are dependent, however, the dependence of x_0 on x_1x_2 is through $y_{01}y_{02}$. If we change $y_{01} = f_{01}(x_0x_1)\,, y_{02} = f_{02}(x_0x_2)$ into $y_{01} = f_{01}(x_0)\,, y_{02} = f_{02}(x_0)$, then, all links become deterministic as in [7,sec.3.5], however, according to (C) the inputs still remain dependent and (B_1) conditions hold. Thus, theorem 1 for two relay Aref network [7] with dependent inputs and (B_1) conditions is as follows:

$$R_a^* = \sup\ min \begin{Bmatrix} H(Y_{23}\,Y_{21}) + H(Y_{01}|X_1) + H(Y_{03}|X_1X_2), \\ H(Y_{13}\,Y_{12}) + H(Y_{02}|X_2) + H(Y_{03}|X_1X_2), \\ H(Y_{13}) + H(Y_{23}) + H(Y_{03}|X_1X_2), \\ H(Y_{01}Y_{02}Y_{03}|X_1X_2) \end{Bmatrix}$$

(b) ***We can validate the theorem 1 by its consistency with the previous results [19],[24],[25]:***

If $X_1 = X_2 = (Y_{02} =)Y_{23} = Y_{13} = Y_{21} = Y_{12} = \emptyset$, then Fig.1 is reduced to broadcast channels with degraded messages, two (three) components and one random component[25],[24],[19] the achievable rate regions of which can be established from the terms in theorem 1(the details are omitted).

Corollary 3: Certain capacity result

The rate R_1^* in theorem 1 coincides with max-flow min-cut upper bound under additional assumptions and results in a certain capacity theorem. More generally, in the main theorem, if we impose the condition ($X_1X_2 \longrightarrow Y_{13}Y_{23} \longrightarrow Y_{03}$, (co-1)) on the random channels distribution A-2 , we will have the following capacity theorem for the network with three random links.(Imposing this condition on A, adds the accompanying condition: ($Y_{01}Y_{02} \longrightarrow X_1X_2 \longrightarrow Y_{03}Y_{13}Y_{23}$, (co-2)). Of course, the conditions (co-1,2) can be replaced by equivalent conditions resulting from the no interference assumption.

Theorem 2: For the special case ($X_1X_2 \longrightarrow Y_{13}Y_{23} \longrightarrow Y_{03}$) the capacity of the network (Fig. 1, $Y_{12} = Y_{21} = \emptyset$) with no interference, random $y_{13}\,, y_{23}\,, y_{03}$ and assuming other links to be deterministic, is given by :

$$C = \sup\ min \begin{Bmatrix} H(Y_{23}) + H(Y_{01}|X_1) + H(Y_{13}|X_1) + H(Y_{03}|Y_{13}Y_{23}) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2), \\ H(Y_{13}) + H(Y_{02}|X_2) + H(Y_{23}|X_2) + H(Y_{03}|Y_{13}Y_{23}) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2), \\ H(Y_{13}) + H(Y_{23}) + H(Y_{03}|Y_{13}Y_{23}) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2), \\ H(Y_{01}Y_{02}Y_{03}Y_{13}Y_{23}|X_1X_2) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2) \end{Bmatrix}$$

Where $y_{01} = f_{01}(x_0\,x_1), y_{02} = f_{02}(x_0x_2)$,
and supremum is taken over the joint distribution of the form : $p(x_0x_1x_2y_{01}y_{02}y_{03}y_{13}y_{23}) =$
$\underbrace{p(x_1x_2)\ p(y_{01}|x_1)p(y_{02}|x_2)\ p(x_0|y_{01}y_{02}x_1x_2)}_{A_2-1}\ \underbrace{p(y_{03}y_{13}y_{23}|x_0x_1x_2)}_{A_2-2}$ (A_2)

A_2-1 and A_2-2 represent equivalent input and random output distributions, respectively. A_2-1 satisfies :
$X_2 \longrightarrow X_1 \longrightarrow Y_{01}\ ,\ Y_{01}X_1 \longrightarrow X_2 \longrightarrow Y_{02}$ (B_2)
Proof: See section IV.

Special cases of theorem 2:

(a) When all of the links are deterministic, using (co-1,2) and the independence of the inputs, theorem 2 gives the capacity of deterministic two relay network as in [7,sec.3.5 by Y_{21} or $Y_{12} = \emptyset$].

(b) If $X_2 = Y_{02} = Y_{23} = \emptyset$, it is readily obtained that theorem 2 gives the capacity of semi-deterministic relay channel in [18].

IV. THE PROOF OF THEOREMS

The proof of the main theorem

Outline of the proof:

As mentioned and explained in subsections B-C in the introduction, we apply decode-and-broadcast strategy. The source x_0 sends the message splits $w_{01}\,, w_{02}$ (sends the corresponding sequences located in bins of $w_{01}\,, w_{02}$)

to the first relay and the second relay, respectively and meanwhile superimposes the message split w_{03} intended for only the receiver over w_{01}, w_{02} . The relays decode what they receive from the source and each other and broadcast the result to the receiver and each other in an appropriate way. The receiver decodes first the messages from the relays and then from the source as will be explained in detail below.

Random Codebook Generation:

The input probability distribution, network transition distribution $p(\cdot\,|\,\cdot)$, $\varepsilon > 0$ and $n \geq 1$ are given. Fix the joint distribution (A) in the main theorem. The idea is to send $w = (w_{03}, w_{01}, w_{02})$ to the receiver.

$\boxed{\text{G1}}$

(G1-a) Generate $2^{nI(U_1;Y_{12})}$ ε-typical independent and identically distribution (i.i.d.) n-sequences $\underline{u}_1 \in \mathcal{U}_1^n$, each with probability :

$$p(\underline{u}_1) = \begin{cases} \dfrac{1}{\|A_\varepsilon^n(U_1)\|} & , \quad \underline{u}_1 \in A_\varepsilon^n(U_1) \\ 0 & , \quad oth. \end{cases}$$

where $\|\cdot\|$ indicates the cardinality. Throw randomly the sequences $\underline{u}_1$ into $2^{nR_{12}}$ bins; index the bins $B_{m_{12}}, m_{12} \in [1, 2^{nR_{12}}], R_{12} < I(U_1; Y_{12})$.

(G1-b) Generate $2^{nI(V_1;Y_{13})}$ ε-typical i.i.d. n-sequences $\underline{v}_1 \in \mathcal{V}_1^n$, each with probability :

$$p(\underline{v}_1) = \begin{cases} \dfrac{1}{\|A_\varepsilon^n(V_1)\|} & , \quad \underline{v}_1 \in A_\varepsilon^n(V_1) \\ 0 & , \quad oth. \end{cases}$$

and throw them randomly into $2^{nR_{13}}$ bins; index the bins $B_{m_{13}}, m_{13} \in [1, 2^{nR_{13}}], R_{13} < I(V_1; Y_{13})$.

(G1-c) We can find jointly ε-typical pairs $(\underline{u}_1, \underline{v}_1)$ such that :

$\big(\underline{u}_1(k_1), \underline{v}_1(j_1)\big) \in (B_{m_{12}} \times B_{m_{13}}) \cap A_\varepsilon^n(U_1, V_1)$, $k_1 \in B_{m_{12}}, j_1 \in B_{m_{13}}$; then for each jointly ε-typical pair $(\underline{u}_1, \underline{v}_1)$, generate one ε-typical conditionally independent n-sequence $\underline{x}_1 \in \mathcal{X}_1^n$ that is jointly ε-typical with that pair, with probability:

$$p(\underline{x}_1) = \begin{cases} \dfrac{1}{\|A_\varepsilon^n(X_1|u_1v_1)\|} & , \quad \underline{x}_1 \in A_\varepsilon^n(X_1U_1V_1) \\ 0 & , \quad oth. \end{cases}$$

and index it as $\underline{x}_1(m_{12}, m_{13}) = \underline{x}_1(s_1)$, $R_{12} + R_{13} < I(U_1; Y_{12}) + I(V_1; Y_{13}) - I(U_1; V_1)$.

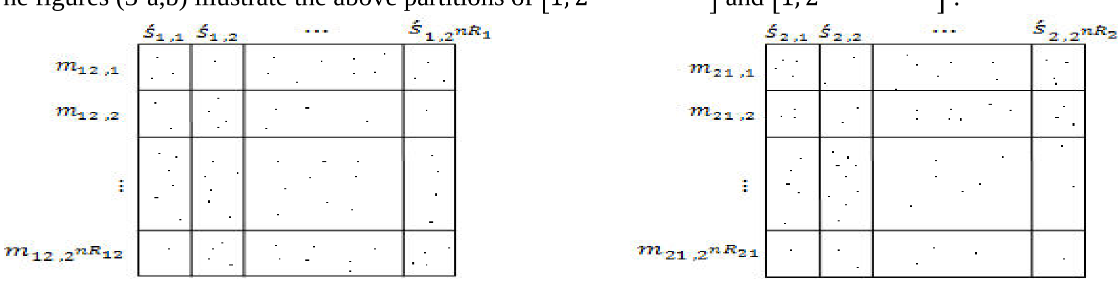
Now, we state briefly the remaining steps of codebook generation and also random partitioning.

(G2-a,b,c) Generate similarly n-sequences $\underline{u}_2$, $\underline{v}_2$, with the bins of $m_{21} \in 2^{nR_{21}}, m_{23} \in 2^{nR_{23}}$, respectively and $\underline{x}_2(m_{21}, m_{23}) = \underline{x}_2(s_2)$, $R_{21} < I(U_2; Y_{21}), R_{23} < I(V_2; Y_{23}), R_{21} + R_{23} < I(U_2; Y_{21}) + I(V_2; Y_{23}) - I(U_2; V_2)$

(G3-a,b,c) Generate, for each $\underline{x}_1(m_{12}, m_{13}) = \underline{x}_1(s_1)$, $2^{nI(U_0;Y_{01}|X_1)}$ $\underline{u}_0$; for each $\underline{x}_2(m_{21}, m_{23}) = \underline{x}_2(s_2)$, $2^{nI(V_0;Y_{02}|X_2)}$ $\underline{v}_0$ with the bins of $w_{01} \in 2^{nR_{01}}, w_{02} \in 2^{nR_{02}}$, respectively, $R_{01} < I(U_0; Y_{01}|X_1), R_{02} < I(V_0; Y_{02}|X_2)$; for each jointly ε-typical pair $(\underline{u}_0, \underline{v}_0)$ (the total number of these pairs is $2^{nR'}, \acute{R} = H(U_0V_0|X_1X_2) - H(U_0|X_1Y_{01}) - H(V_0|X_2Y_{02}) > R_{01} + R_{02}$) $2^{nR_{03}}$ $\underline{x}_0(w_{03}|w_{01}w_{02}s_1s_2)$.

Random Partitions: Randomly partition $\underline{u}_0$ sequences into $2^{nR_{12}} \times 2^{n\acute{R}_1}$, $\underline{v}_0$ sequences into $2^{nR_{21}} \times 2^{n\acute{R}_2}$, $(\underline{u}_0, \underline{v}_0)$ pairs into $2^{n(R_{01}+R_{02})}$ disjoint cells; $\acute{R}_1 = R_{01} - R_{12}, \acute{R}_2 = R_{02} - R_{21}$.Therefore, $\forall\, \underline{u}_0 \in B_{w_{01}} \longrightarrow w_{01} \equiv (\acute{s}_1, m_{12}), \acute{s}_1 \in [1, 2^{n\acute{R}_1}]$ and not vice versa. Similarly, $\forall\; \underline{v}_0 \in B_{w_{02}} \longrightarrow w_{02} \equiv (\acute{s}_2, m_{21})$, $\acute{s}_2 \in [1, 2^{n\acute{R}_2}]$ and not vice versa. And also, $m_{13} \equiv (\acute{s}_1, u_{12}), \acute{s}_1 \in [1, 2^{n\acute{R}_1}], u_{12} \in [1, 2^{nR_{u_{12}}}], R_{13} = \acute{R}_1 + R_{u_{12}}$, $R_{u_{12}} = \alpha(R_{12} + R_{21}), 0 \leq \alpha \leq 1, m_{23} \equiv (\acute{s}_2, u_{21}), \acute{s}_2 \in [1, 2^{n\acute{R}_2}], u_{21} \in [1, 2^{nR_{u_{21}}}], R_{23} = \acute{R}_2 + R_{u_{21}}$, $R_{u_{21}} = \overline{\alpha}(R_{21} + R_{12}), \overline{\alpha} = 1 - \alpha, (u_{12}, u_{21}) = (m_{12}, m_{21})$.

The figures (3-a,b) illustrate the above partitions of $[1, 2^{nI(U_0;Y_{01}|X_1)}]$ and $[1, 2^{nI(V_0;Y_{02}|X_2)}]$:



(a) (b)

Fig3.Random Partitions (a) for $\underline{u}_0$ sequences (b) for $\underline{v}_0$ sequences

Encoding:

Let $w_i = (w_{03,i}, w_{01,i}, w_{02,i})$ be the message to be sent in block i and assume that at the end of block $i-1$: The first relay has decoded $w_{01,i-1}, m_{21,i-1}$ and knows $m_{12,i-1}$. The second relay has decoded $w_{02,i-1}, m_{12,i-1}$ and knows $m_{21,i-1}$. Therefore, either relays know $(m_{12,i-1}, m_{21,i-1})$ and can send this common information to the receiver time-sharingly, that is, the first relay sends $u_{12,i}$ and the second relay sends $u_{21,i}$, thereby, the receiver can decode $(u_{12,i}, u_{21,i}) = (m_{12,i-1}, m_{21,i-1})$, $R_{u_{12}} = \alpha(R_{12}+R_{21})$, $0 \leq \alpha \leq 1$, $R_{u_{21}} = \overline{\alpha}(R_{21}+R_{12})$, $\overline{\alpha} = 1-\alpha$.

Hence, at the beginning of block i : The first relay sends $\underline{x}_1(m_{12,i}, m_{13,i}) = \underline{x}_1(s_{1,i})$, the second relay sends $\underline{x}_2(m_{21,i}, m_{23,i}) = \underline{x}_2(s_{2,i})$, the transmitter sends $\underline{x}_0(w_{03,i}|w_{01,i}, w_{02,i}, s_{1,i}, s_{2,i})$.

Decoding and probability of error analysis: We defer the details of decoding and probability of error analysis to the Appendix. The analysis shows that if the rate R^* is as in the statement of the main theorem, the reliable communication is possible.

The proof of theorem 1:

Assuming that the transmitter knows the first symbols of $\underline{x}_1$ and $\underline{x}_2$ sequences and noting to the deterministic functions: $y_{01i} = f_{01}(x_{0i}, x_{1i})$, $y_{02i} = f_{02}(x_{0i}, x_{2i})$, $y_{12i} = f_{12}(x_{1i})$, $y_{21i} = f_{21}(x_{2i})$, $y_{13i} = f_{13}(x_{1i})$, $y_{23i} = f_{23}(x_{2i})$, $x_{1i} = f_{1i}(y_{01}^{i-1}, y_{21}^{i-1}) = f_{1i}(y_{011}, y_{012}, \cdots, y_{01,i-1}; y_{211}, \cdots, y_{21,i-1})$, $x_{2i} = f_{2i}(y_{02}^{i-1}, y_{12}^{i-1}) = f_{2i}(y_{021}, \cdots, y_{02,i-1}; y_{121}, \cdots, y_{12,i-1})$, $i = 1, \cdots, n$, we conclude that the transmitter knows $\underline{y}_{01}$, $\underline{y}_{02}$, $\underline{y}_{12}$, $\underline{y}_{21}$, $\underline{y}_{13}$ and $\underline{y}_{23}$, hence, these sequences can be generated at the transmitter or can be treated as equivalent input random variables.Therefore,by putting $U_0 = Y_{01}, V_0 = Y_{02}, U_2 = Y_{21}, U_1 = Y_{12}, V_2 = Y_{23}, V_1 = Y_{13}$ in the main theorem ([19,theorem 3],[26],[27]) and noting to deterministic functions, R_1^* is obtained from R^* . ■

The proof of theorem 2:

Achievability : As explained in the proof of theorem 1, in this case, we can put in the main theorem $U_0 = Y_{01}$, $V_0 = Y_{02}$, $V_2 = X_2$, $V_1 = X_1$, $Y_{12} = Y_{21} = U_1 = U_2 = \emptyset$, and have :

(i) The first term in $R^* \longrightarrow$

$$H(Y_{01}|X_1) + H(Y_{23}) - H(Y_{23}|X_2) + H(Y_{03}Y_{13}Y_{23}|X_1X_2) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2)$$
$$\stackrel{e}{=} H(Y_{01}|X_1) + H(Y_{23}) + H(Y_{13}|X_1) + H(Y_{03}|Y_{13}Y_{23}) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2) ,$$

(ii) The second term in $R^* \xrightarrow{f} H(Y_{02}|X_2) + H(Y_{13}) + H(Y_{23}|X_2) + H(Y_{03}|Y_{13}Y_{23}) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2)$,

(iii) The third term in $R^* \xrightarrow{g} H(Y_{13}) + H(Y_{23}) + H(Y_{03}|Y_{13}Y_{23}) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2)$,

(iv) The fourth term in $R^* \xrightarrow{h} H(Y_{03}Y_{13}Y_{23}Y_{01}Y_{02}|X_1X_2) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2)$,

where e,f, and g follow from the intropy relations, the conditional independence of $Y_{13}Y_{23}$ and the assumption $X_1X_2 \longrightarrow Y_{13}Y_{23} \longrightarrow Y_{03}$ in theorem 2, h follows from $U_0 = Y_{01} = f_{01}(X_0X_1)$ and $V_0 = Y_{02} = f_{02}(X_0X_2)$.

Converse : Using max flow-min cut upper bound in [23] for Fig.1 and (a) entropy and information relations ($H(X,Y) = H(X) + H(Y|X), H(A|B,C) \leq H(A|B), I(X;Y) = H(Y) - H(Y|X)$), (b) deterministic functions $y_{01} = f_{01}(x_0x_1), y_{02} = f_{02}(x_0x_2)$, we can prove:

(i) $I(X_0X_2; Y_{01}Y_{03}Y_{23}Y_{13}|X_1) \stackrel{(a),(b)}{=} \cdots \stackrel{(a)}{\leq} H(Y_{01}|X_1) + H(Y_{23}) + H(Y_{13}|X_1) + H(Y_{03}|Y_{13}Y_{23}) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2)$.

Similarly, the other three conditions are proved (for brevity, the details are omitted):

(ii) $I(X_0X_1; Y_{02}Y_{03}Y_{13}Y_{23}|X_2) \leq \cdots = H(Y_{13}) + H(Y_{02}|X_2) + H(Y_{23}|X_2) + H(Y_{03}|Y_{13}Y_{23}) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2)$,

(iii) $I(X_0X_1X_2; Y_{03}Y_{13}Y_{23}) = \cdots \leq H(Y_{13}) + H(Y_{23}) + H(Y_{03}|Y_{13}Y_{23}) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2)$,

(iv) $I(X_0; Y_{01}Y_{03}Y_{02}Y_{13}Y_{23}|X_1X_2) = \cdots = H(Y_{01}Y_{02}Y_{03}Y_{13}Y_{23}|X_1X_2) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2)$.

The above proof completes the converse. ■

V. CONCLUSION

A new achievable rate as a main theorem for a stochastic two relay network with no interference was established which gives the lower bound for the general relay channel obtained by Cover and El Gamal with slight difference due to no interference assumption at the receiver , includes the one relay rates of a two-level relay channel studied by Gupta and Kumar using point to point coding, includes the rates of two relay Aref network and other special two relay networks, meets the max-flow min-cut upper bound under certain additional assumptions resulting in certain capacity theorem which includes the related previous capacities and is validated by its consistency with previous results relevant to special cases of broadcast channels. The theorems might be generalized to networks having more than two relays.

APPENDIX
Decoding and Probability of Error Analysis

(D-1) The first relay, knowing $s_{1,i}$ and upon receiving $\underline{y}_{01}$, finds $\underline{u}_0$: $\left(\underline{u}_0, \underline{y}_{01}\right) \in A_\varepsilon^n$ and then finds the bin of $\underline{u}_0$, that is, $\widehat{w}_{01,i}$ and declares with small probability of error: $\widehat{w}_{01,i} = w_{01,i}$ iff n is sufficiently large and : [19],[20],[21]

$$R_{01} < I(U_0;Y_{01}|X_1) \qquad \text{(1-a)}$$

And the second relay, knowing $s_{2,i}$ and upon receiving $\underline{y}_{02}$, finds $\underline{v}_0$: $\left(\underline{v}_0, \underline{y}_{02}\right) \in A_\varepsilon^n$ and then finds the bin of $\underline{v}_0$, that is, $\widehat{w}_{02,i}$ and declares with small probability of error: $\widehat{w}_{02,i} = w_{02,i}$ iff n is sufficiently large and

$$R_{02} < I(V_0;Y_{02}|X_2) \qquad \text{(1-b)}$$

And there are unique $\left(\widehat{w}_{01,i}\ , \widehat{w}_{02,i}\right) = \left(w_{01,i}\ , w_{02,i}\right)$ iff :

$R_{01} + R_{02} < I(U_0;Y_{01}|X_1) + I(V_0;Y_{02}|X_2) - I(U_0;V_0|X_1X_2) \overset{(A)}{=} H(U_0V_0|X_1X_2) - H(U_0|X_1Y_{01}) - H(V_0|X_2Y_{02})$ (1-c)

(D-2) The second relay and the receiver act as two receivers for broadcasted $\underline{x}_1$ .The receiver, upon receiving $\underline{y}_{13}$ and finding $\underline{v}_1$:$\left(\underline{v}_1, \underline{y}_{13}\right) \in A_\varepsilon^n$, and the bin of $\underline{v}_1$,that is, $\widehat{\widehat{m}}_{13,i}$, declares $\widehat{\widehat{m}}_{13,i} = m_{13,i}$ ($\hat{\hat{\acute{s}}}_{1,i} = \acute{s}_{1,i}$, $\hat{\hat{u}}_{12,i} = u_{12,i}$) and the second relay, upon receiving $\underline{y}_{12}$ and finding $\underline{u}_1$:$\left(\underline{u}_1, \underline{y}_{12}\right) \in A_\varepsilon^n$ and its bin number $\widehat{m}_{12,i}$, declares $\widehat{m}_{12,i} = m_{12,i}$, all with small probability of error iff n is sufficiently large and: [19],[20],[21]

$$R_{13} = R'_1 + \alpha(R_{12} + R_{21}) < I(V_1;Y_{13})\ , \qquad \alpha \in [0,1] \qquad \text{(2-a)}$$

$$R_{12} < I(U_1;Y_{12}) - I(U_1;V_1) \qquad \text{(2-b)}$$

(D-3) The first relay and the receiver act as two receivers for broadcasted $\underline{x}_2$. The receiver, upon receiving $\underline{y}_{23}$ and finding $\underline{v}_2$:$\left(\underline{v}_2, \underline{y}_{23}\right) \in A_\varepsilon^n$, and the bin of $\underline{v}_2$,that is, $\widehat{\widehat{m}}_{23,i}$, declares $\widehat{\widehat{m}}_{23,i} = m_{23,i}$ ($\hat{\hat{\acute{s}}}_{2,i} = \acute{s}_{2,i}$, $\hat{\hat{u}}_{21,i} = u_{21,i}$) and the second relay, upon receiving $\underline{y}_{21}$ and finding $\underline{u}_2$:$\left(\underline{u}_2, \underline{y}_{21}\right) \in A_\varepsilon^n$ and its bin number $\widehat{m}_{21,i}$, declares $\widehat{m}_{21,i} = m_{21,i}$, all with small probability of error iff n is sufficiently large and: [19],[20],[21]

$$R_{23} = R'_2 + \overline{\alpha}(R_{21} + R_{12}) < I(V_2;Y_{23})\ , \qquad \overline{\alpha} = 1 - \alpha \qquad \text{(3-a)}$$

$$R_{21} < I(U_2;Y_{21}) - I(U_2;V_2) \qquad \text{(3-b)}$$

(D-4) The receiver, upon receiving $\underline{y}_{03}$ and having received $\left(\underline{y}_{13}, \underline{y}_{23}\right)$, found $\hat{\hat{u}}_{12,i-1}$, $\hat{\hat{u}}_{21,i-1}$, $\hat{\hat{\acute{s}}}_{1,i-1}$, $\hat{\hat{\acute{s}}}_{2,i-1}$, $\widehat{\widehat{m}}_{12,i-1}$, $\widehat{\widehat{m}}_{21,i-1}$ and hence knowing $\hat{\hat{s}}_{1,i-1}$, $\hat{\hat{s}}_{2,i-1}$, finds jointly typical pair $\left(\underline{u}_0\ , \underline{v}_0\right) \in L\left(\underline{y}_{03,i-2}\ , \underline{y}_{13,i-2}\ , \underline{y}_{23,i-2}\right) \cap (A_1 \times B_1) \cap A_\varepsilon^n(U_0V_0|X_1X_2)$ and then $\left(\widehat{\widehat{w}}_{01,i-2}\ , \widehat{\widehat{w}}_{02,i-2}\right)$ with small probability of error iff n is sufficiently large and (see random partition) :

$$\acute{R} < R_{01} + R_{02} + I(U_0V_0;Y_{03}Y_{13}Y_{23}|X_1X_2) \qquad \text{(4)}$$

Where

$$L\left(\underline{y}_{03,i-2}\ , \underline{y}_{13,i-2}\ , \underline{y}_{23,i-2}\right) = \left\{\left(\underline{u}_0\ , \underline{v}_0\right) : \left(\underline{u}_0\ , \underline{v}_0\ , \underline{x}_1\ , \underline{x}_2\ , \underline{y}_{03}\ , \underline{y}_{13}\ , \underline{y}_{23}\right) \in A_\varepsilon^n\right\}\ ,$$

$$A_1 = \left\{\underline{u}_0 : \underline{u}_0 \in S_{1,\hat{\hat{s}}_{1,i-1}} \cap S_{12,\widehat{\widehat{m}}_{12,i-1}} \cap A_\varepsilon^n(U_0|X_1X_2)\right\}\ ,\ B_1 = \left\{\underline{v}_0 : \underline{v}_0 \in S_{2,\hat{\hat{s}}_{2,i-1}} \cap S_{21,\widehat{\widehat{m}}_{21,i-1}} \cap A_\varepsilon^n(V_0|X_1X_2)\right\}$$

(D-5) The receiver, upon receiving $\underline{y}_{03}$ and having found jointly typical pair $\left(\underline{u}_0\ , \underline{v}_0\right)$ or $\left(\widehat{\widehat{w}}_{01,i-2}\ , \widehat{\widehat{w}}_{02,i-2}\right)$ and knowing $\left(\underline{x}_1\ , \underline{x}_2\right)$, $\left(\underline{y}_{13}, \underline{y}_{23}\right)$ finds $\underline{x}_0$ such that $\left(\underline{x}_0\ , \underline{y}_{03}\ , \underline{y}_{13}\ , \underline{y}_{23}\ , \underline{x}_1\ , \underline{x}_2\ , \underline{u}_0\ , \underline{v}_0\right) \in A_\varepsilon^n$ and declares $\widehat{w}_{03,i-2} = w_{03,i-2}$ with small probability of error iff n is sufficiently large and :

$$R_{03} < I(X_0;Y_{03}Y_{13}Y_{23}|X_1X_2U_0V_0) \qquad \text{(5)}$$

Now, we are able to find bounds on the total rate R for the message $w = (w_{01}\ , w_{02}\ , w_{03})$. From code construction and (G-3) in codebook generation , we have:

$$R = \underbrace{R_{01} + R_{02}} + R_{03} < R' + R_{03} \qquad \text{(6)}$$

$$\text{(4,5,6)} \Longrightarrow R < \underbrace{R_{01} + R_{02}} + H(Y_{03}Y_{13}Y_{23}|X_1X_2) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2) \qquad \text{(7)}$$

(B-1)

(2-a) + (3-a) $\Rightarrow$ $R'_1 + R_{12} + R'_2 + R_{21} < I(V_1;Y_{13}) + I(V_2;Y_{23}), \quad 0 < \alpha < 1$ (8)

(B-2)

$$R'_1 + R_{12} = \underbrace{R'_1 + \alpha R_{12}} + \underbrace{\overline{\alpha} R_{12}} \overset{(2-a,b)}{<} I(V_1;Y_{13}) + I(U_1;Y_{12}) - I(U_1;V_1), \ \ \alpha = 0 \qquad (9\text{-a})$$

$$R'_2 + R_{21} = \underbrace{R'_2 + \alpha R_{21}} + \underbrace{\overline{\alpha} R_{21}} \overset{(3-a,b)}{<} I(V_2;Y_{23}) + I(U_2;Y_{21}) - I(U_2;V_2), \ \ \alpha = 1 \qquad (9\text{-b})$$

(B-3) From random coding and random partitioning, we have:

$$R'_1 + R_{12} = R_{01} \qquad (10\text{-a})$$

$$R'_2 + R_{21} = R_{02} \qquad (10\text{-b})$$

$$R' = H(U_0V_0|X_1X_2) - H(U_0|X_1Y_{01}) - H(V_0|X_2Y_{02}) \qquad (10\text{-c})$$

(B-4)

$$R \overset{(7),(1,10-a),(9-b)}{<} I(U_0;Y_{01}|X_1) + I(V_2;Y_{23}) + I(U_2;Y_{21}) - I(U_2;V_2) + H(Y_{03}Y_{13}Y_{23}|X_1X_2) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2) = R^*_{c_1} \qquad (11\text{-a}),$$

and

$$R \overset{(7),(1,10-b),(9-a)}{<} I(V_0;Y_{02}|X_2) + I(V_1;Y_{13}) + I(U_1;Y_{12}) - I(U_1;V_1) + H(Y_{03}Y_{13}Y_{23}|X_1X_2) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2) = R^*_{c_2} \qquad (11\text{-b}),$$

and

$$R \overset{(7),(8)}{<} I(V_1;Y_{13}) + I(V_2;Y_{23}) + H(Y_{03}Y_{13}Y_{23}|X_1X_2) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2) = R^*_{c_3} \qquad (11\text{-c}),$$

and

$$R \overset{(5),(10-c)}{<} H(Y_{03}Y_{13}Y_{23}V_0U_0|X_1X_2) - H(Y_{03}Y_{13}Y_{23}|X_0X_1X_2) - H(U_0|X_1Y_{01}) - H(V_0|X_2Y_{02}) = R^*_{c_4} \qquad (11\text{-d}),$$

We know that:

The average probability of error is small arbitrarily $\Leftrightarrow$ decoding steps (D-1) – (D-5) and the corresponding bounds are satisfied $\Rightarrow$ the bounds (11-a, b, c, d) are satisfied, or equivalently:

At least one of the bounds (11-a, b, c, d) is not satisfied $\Rightarrow$ at least one of the decoding steps is not satisfied $\Leftrightarrow$ the average probability of error is not small arbitrarily.

So, The bounds (11-a, b, c, d) complete the proof of the main theorem. ■

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