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VAN-KAMPEN TYPE THEOREM FOR TOPOLOGICAL FUNDAMENTAL GROUPS

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ABSTRACT. In this talk we present a Van-Kampen type theorem for topological fundamental groups. As a consequence we show that the topological fundamental group of a union space of open subspaces with simply connected mutually intersections is a coproduct in the category of topological groups.

1. Introduction and Motivation

Historically, J. Dugundji in 1950, put, for the first time, a topology on fundamental groups of certain spaces and deduced a classification theorem for connected covers of a space.

Recently, Biss [1] generalized the results announced by J. Dugundji. He equipped the fundamental group of a pointed space (X, x) with the quotient topology inherited from $Hom((S^1, 1), (X, x))$ with compact-open topology and denoted by $\pi_1^{top}(X, x)$. He proved among other things that $\pi_1^{top}(X, x)$ is a topological group which is independent of the base point in path components and π_1^{top} is a functor from the homotopy category of based spaces to the category of topological groups which preserves the direct product. He showed that π_1^{top} is discrete if and only if the space X is semilocally simply connected. However, P. Fabel [3] mentioned that path connectedness and locally path connectedness of X is necessary.

In this note, we study on the structure of a topological fundamental group of a space which is covered by a family of path connected open subspaces satisfying in some conditions (see Theorem 3.2). Of course, in this case there exists a famous theorem, called Van-Kampen theorem which clarifies the algebraic structure of the

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fundamental group; But In this talk, by the notion of Biss, we deduce a new type of Van-Kampen Theorem for both algebraic and topological structure of topological fundamental groups.

2. Van-Kampen Theorem

Suppose that X is a path connected topological space and $x_0 \in X$ is arbitrary. Let $\{U_\lambda; \lambda \in \Lambda\}$ be a cover of X by path connected open sets such that for all $\lambda \in \Lambda$, $x_0 \in U_\lambda$ and for any $\lambda_1, \lambda_2 \in \Lambda$, there exists $\lambda' \in \Lambda$ with $U_{\lambda_1} \cap U_{\lambda_2} = U_{\lambda'}$.

If $U_\lambda \subseteq U_\mu$, the notation $\phi_{\lambda\mu} : \pi_1(U_\lambda) \to \pi_1(U_\mu)$ is the homomorphism induced by the inclusion. The notation $\psi_\lambda : \pi_1(U_\lambda) \to X$ is also the homomorphism induced by the inclusion. Then the following diagram is commutative:

$$\begin{array}{ccc} \pi_1(U_\lambda) & & \\ {\scriptstyle \phi_{\lambda\mu}}\downarrow & \searrow{\scriptstyle \psi_\lambda} & \\ \pi_1(U_\mu) & \xrightarrow{\psi_\mu} & \pi_1(X). \end{array}$$

Theorem 2.1. ([5]) *Under the above hypothesis, the group $\pi_1(X)$ satisfies the following universal condition: Let H be any group and $\rho_\lambda : \pi_1(U_\lambda) \to H$ be any collection of homomorphisms defined for all $\lambda \in \Lambda$ such that if $U_\lambda \subseteq U_\mu$, then the following diagram is commutative:*

$$\begin{array}{ccc} \pi_1(U_\lambda) & & \\ {\scriptstyle \phi_{\lambda\mu}}\downarrow & \searrow{\scriptstyle \rho_\lambda} & \\ \pi_1(U_\mu) & \xrightarrow{\rho_\mu} & H. \end{array}$$

Then there exists a unique homomorphism $\sigma : \pi_1(X) \to H$ with the following commutative diagram:

$$\begin{array}{ccc} \pi_1(U_\lambda) & \xrightarrow{\psi_\lambda} & \pi_1(X) \\ & {\scriptstyle \rho_\lambda}\searrow & \downarrow{\scriptstyle \sigma} \\ & & H. \end{array}$$

In the following we give a sketch of proof of the theorem and refer to [5] for more details.

Proof. First, by a lemme [5], we note that the group $\pi_1(X)$ is generated by the union of the images $\psi[\pi_1(U_\lambda)]$'s. Hence for any $\alpha \in \pi_1(X)$, we have $\alpha = \psi_{\lambda_1}(\alpha_1)\psi_{\lambda_2}(\alpha_2)\cdots\psi_{\lambda_n}(\alpha_n)$ $(*)$ where $\alpha_i \in \pi_1(U_{\lambda_i})$; and so the homomorphism σ, if it exists, must be unique and defined as follows

$$\sigma(\alpha) = \rho_{\lambda_1}(\alpha_1)\rho_{\lambda_2}(\alpha_2)\cdots\rho_{\lambda_n}(\alpha_n).$$

It has been proved that the definition of σ is independent of the presentation in the form $(*)$. □

3. Van-Kampen type Theorem for topological fundamental groups

In this section, we want to prove our main result. Under the hypothesis mentioned in the previous section, we establish the Van-Kampen Theorem for topological fundamental groups as follows:

Theorem 3.1. *The topological group $\pi_1^{top}(X)$ satisfies the universal property in the category of topological groups; that is for any topological group H and any collection of continuous homomorphisms $\rho_\lambda : \pi_1^{top}(U_\lambda) \to H$ defined for all $\lambda \in \Lambda$ so that if $U_\lambda \subseteq U_\mu$, the following diagram is commutative:*

$$\begin{array}{ccc} \pi_1^{top}(U_\lambda) & & \\ {\scriptstyle \phi_{\lambda\mu}}\downarrow & \searrow{\scriptstyle \rho_\lambda} & \\ \pi_1^{top}(U_\mu) & \xrightarrow{\rho_\mu} & H. \end{array}$$

Then there exists a unique continuous homomorphism $\sigma : \pi_1^{top}(X) \to H$ with the following commutative diagram:

$$\begin{array}{ccc} \pi_1^{top}(U_\lambda) & \xrightarrow{\psi_\lambda} & \pi_1^{top}(X) \\ & {\scriptstyle \rho_\lambda}\searrow & \downarrow{\scriptstyle \sigma} \\ & & H. \end{array}$$

Proof. First we recall that the homomorphism σ which is introduced in the proof of Van-Kampen Theorem. In this case, by the definition of this map and the fact that ρ_λ are continuous, we can prove that σ is also continuous. Hence we can rewrite Van-Kampen Theorem for both algebraic and topological structure, as above. □

In the following we recall the definition of teh free topological product of topological groups [6].

Definition 3.2. Let $\{G_i; i \in I\}$ be a set of topological groups. Then the topological group F is said to be the free topological product of G_is, if it has the following properties:

i) for each $i \in I$, G_i is a subgroup of F;

ii) F is algebraically generated by $\cup G_i$;

iii) if for each topological group H, $\phi_i : G_i \to H$ is a continuous homomorphism, for any $i \in I$, then there exists a continuous homomorphism ϕ of F into H such that $\phi = \phi_i$ on G_i, for each i.

The following corollary is an interesting consequence of theorem 3.1 according to the above notation of free topological groups.

Corollary 3.3. *Further to the assumption of the previous theorem, suppose that for any $\lambda, \mu \in \Lambda$, $U_\lambda \cap U_\mu$ is simply connected. Then the topological fundamental group of $X = \cup_{\lambda\in\Lambda} U_\lambda$ is the free topological product of $\pi_1^{top}(U_\lambda)$'s.*

Remark 3.4. The notion of topological fundamental group has been extended to the topological homotopy groups [4]. By this new notion, we can generalize our main result to a Van-Kampen type theorem for topological homotopy groups.

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