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WEAK-KADEC RENORMABLE BANACH SPACES

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ABSTRACT. We will use a game theoretic proof to show that if a Banach space X admits an equivalent weak-Kadec norm, then $(X, weak)$ is σ-fragmented by the norm.

1. Introduction.

Let X be a Banach space. A norm $||\ .\ ||$ on X is said to be *Kadec* if weak and norm topologies coincide on the unit sphere of X.
Following [1], a topological space (X, τ) is said to be *σ-fragmented* by a metric ρ if for each $\epsilon > 0$, X can be written as $X = \bigcup_{n=1}^{\infty} X_{n,\epsilon}$ such that for each $n \geq 1$ and nonempty set $A \subset X_{n,\varepsilon}$, there is a relatively τ-open nonempty subset $B \subset A$ of ρ-diameter less then ϵ. Kenderov and Moors in [4] used the following topological game to characterize σ-fragmentability of a topological space X:
Two players Σ and Ω alternatively select non-empty subsets of X. Σ usually starts the game by choosing some non-empty subset A_1 of X. Then Ω selects some non-empty relatively open subset B_1 of A_1. On the n-th stage of the game, Σ takes a non-empty subset A_n of the last move B_{n-1} of Ω and the latter answers by taking again a relatively open subset B_n of A_n. By continuing this procedure, the two players generate a sequence of sets $p = (A_i, B_i)_{i=1}^{\infty}$, which is called a play. A *strategy for the Ω-player,* is a rule which determines Ω's move at each stage based on the game played so far following the strategy.

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Theorem 1.1. ([4], **Theorems 1.3, 1.4 and 2.1**) *For a Banach space X the following are equivalent:*
(i) (X, weak) is σ-fragmented by a metric which is stronger than the weak topology.
(ii) There exists a strategy s for the player Ω such that, for every s-play $p = (A_i, B_i)_{i\geq 1}$ either $\bigcap_{i\geq 1} B_i = \emptyset$ or $\lim_{i\to\infty}$ norm-diam $(B_i) = 0$.

Moreover, we have the following:

Theorem 1.2. ([2], **Theorem 2**) *Let (X, τ) be a topological space and d be a metric on it which is stronger than the τ-topology. If X is σ-fragmented by d, then it is σ-fragmented by d using d-closed sets.*

2. Results

We begin with the following lemma:

Lemma 2.1. *Let G be a weak open subset of a Banach space X; $\delta, \epsilon > 0$; $x \in G \cap \{z : ||z|| = \delta\}$ and the norm diameter of $G \cap \{z : ||z|| = \delta\}$ is less than $\epsilon/2$. Then there exists a neighborhood W of x and a positive real number $\alpha_{\epsilon,x}$, such that $y \in W$ and $||y|| - \delta < \alpha_{\epsilon,x}$ implies that $||x - y|| < \epsilon$.*

Proof. Let U be a neighborhood of 0 such that $U + U \subset G \setminus \{x\}$. Take $\epsilon/2 > \alpha_{\epsilon,x} > 0$ such that $\alpha_{\epsilon,x} B \subset U$. Define $W = (U + \{x\}) \setminus (\delta - \alpha_{\epsilon,x})B$. If $y \in W$ and $||y|| < \delta + \alpha_{\epsilon,x}$, then $|||y|| - \delta| < \alpha_{\epsilon,x}$, since $y \notin (\delta - \alpha_{\epsilon,x})B$. It follows that

$$||(\delta/||y||)y - y|| = |\delta - ||y||| < \alpha_{\epsilon,x}.$$

Hence $(\delta/||y||)y = (\delta/||y||)y - y + y \in U + U + \{x\} \subset G$. Thus $(\delta/||y||)y \in G \cap \{z : ||z|| = \delta\}$. Therefore $||(\delta/||y||)y - x|| < \epsilon/2$. It follows that

$$||y - x|| < ||y - (\delta/||y||)y|| + ||(\delta/||y||)y - x|| < \epsilon/2 + \epsilon/2 = \epsilon. \square$$

We also need the following result:

Lemma 2.2. ([4], **Proposition 2.1**) *If the unit ball B of a Banach space X admits a strategy s with the equivalent properties of Theorem 1.1, then the whole space also admits such a strategy.*

Theorem 2.3. *Let X be a Kadec renormable Banach space. Then $(X, weak)$ is σ-fragmentable.*

Proof. According to Theorem 1.1 and Lemma 2.2, it is enough to show that in $(B, weak)$, where B denotes the unit ball of X, the player Ω has a winning strategy which satisfies one of the equivalent properties of Theorem 1.1. We use the terminologies of Lemma 2.1
Let $||\ .\ ||$ denote an equivalent Kadec norm on X and $A_1 \subset B$ be the first choice of Σ-player. Put

$$\rho_1 = \sup\{||x|| : x \in A_1\} \text{ and } \epsilon_1 = 1.$$

Two cases may happen.
(1) There is an element $x_1 \in A_1$ such that $\alpha_{\epsilon_1,x_1} + ||x_1|| > \rho_1$. Then we take such a point x_1 and we define $s_1(A_1) = B_1 = W_{\epsilon_1,x_1} \cap A_1$ and $\epsilon_2 = \epsilon_1/2$. Then for each $y \in B_1$, $||y|| \leq \rho_1 < \alpha_{\epsilon_1,x_1} + ||x_1||$. Therefore, by Lemma 2.1, $||y - x_1|| < \epsilon_1$. Hence $||\ ||-diam(B_1) < 2.\epsilon_1$.
(2) For every $x \in A_1, \alpha_{\epsilon_1,x} \leq \rho_1$.
In this case, take some point $x_1 \in A_1$ such that its norm is bigger than $(1/2)\rho_1$. Then define

$$s_1(A_1) = B_1 = W_{\epsilon_1,x_1} \cap A_1 \setminus (1/2)\rho_1 B \text{ and } \epsilon_2 = \epsilon_1.$$

Let $(A_i, B_i)_{1\leq i\leq n}$; $\{\epsilon_i\}_{1\leq i\leq n}$, and $x_1, \ldots, x_n$ have already been selected. If A_{n+1} is the next move of Σ-player and

$$\rho_{n+1} = sup\{||x|| : x \in A_{n+1}\},$$

then we consider the following two possible cases.
(1) There exists an element $x_{n+1} \in A_{n+1}$, such that $\alpha_{\epsilon_{n+1},x_{n+1}} + ||x_{n+1}|| > \rho_{n+1}$. In this case, we take such a point x_{n+1} and we define

$$s_{n+1}(A_1, \ldots, A_{n+1}) = B_{n+1} = W_{\epsilon_{n+1},x_{n+1}} \cap A_{n+1} \text{ and } \epsilon_{n+2} = \epsilon_{n+1}/2$$

(2) For every point $x \in A_{n+1}, \alpha_{\epsilon_{n+1},x} + ||x|| \leq \rho_{n+1}$.
In this case, we take some $x_{n+1} \in A_{n+1}$ with $||x_{n+1}|| > (1 - 1/(n+1))\rho_{n+1}$, then we define

$$s_{n+1}(A_1, \ldots, A_{n+1}) = B_{n+1} = W_{\epsilon_{n+1},x_{n+1}} \cap A_{n+1} \setminus (1 - \frac{1}{(n+1)})\rho_{n+1} B$$

and $\epsilon_{n+2} = \epsilon_{n+1}$. Therefore, the strategy $s = \{s_n\}$ for the Ω-player inductively is defined.
If $x \in \bigcap_{n\geq 1} A_n$ and $\lim_{n\to\infty} ||\ .\ ||-diam(B_n) \neq 0$, then there exists some $\epsilon > 0$, such that $||\ .\ ||-diam(B_n) > \epsilon$ for each $n \in N$. This means that for all but finitely many n, the case (2) happens and thus

$\{\epsilon_n\}$ is eventually constant. We may suppose that for all n, $\epsilon < 2\epsilon_n$. Since $x \in \bigcap_{n\geq 1} A_n$,

$$(1 - \frac{1}{n})\rho_n < ||x|| < \rho_n, \ \forall n.$$

Let $\rho_n \searrow \rho$. Then the above inequality shows that $||x|| = \rho$. Take $W_{\epsilon/2,x}$, then $\alpha_{\epsilon/2,x} + ||x| > ||x| = \lim_{n\to\infty} \rho_n$. Therefore, there is some $n_0 \in N$, such that

$$\forall n \geq n_0, \ \rho_n < \alpha_{\epsilon/2,x} + ||x|| \leq \alpha_{\epsilon_n,x} + ||x||.$$

Since $x \in A_n$ for each n, this is a contradiction.□

References

[1] J. E. Jayne, I. Namioka and C. A. Rogers, *Topological properties of Banach spaces*, Proc. London Math. Soc. (3) 66(1993), 651-672.

[2] J. E. Jayne, I. Namioka and C. A. Rogers, *σ-Fragmentable Banach spaces* I, Mathematika, 39(1992), 161-188

[3] P. S. Kenderov and W. B. Moors, *Game characterization of fragmentability of topological spaces*, Mathematics and Education in Mathematics, (1996) 8-18 *(Proceedings of the 25-th Spring conference of the Union of Bulgarian Mathematicians, April 1996, Kazanlak, Bulgaria).*

[4] P. S. Kenderov and W. B. Moors, *Fragmentability and sigma-fragmentability of Banach spaces* J. London Math. Soc. (2) 60 (1999) 203-223.