Some Properties of n-Isoclinism in Lie Algebras

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Abstract

In 1940, P. Hall introduced the concept of isoclinism of groups and it was generalized to n-isoclinism and isologism with respect to a given variety of groups.

In the present article this notion is studied in Lie algebras and give some results similar to N.S. Hekster in 1986. In particular, it is shown that every family of n-isoclinism of Lie algebras contains an n-stem Lie algebra of minimal dimension. [1]

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1. Introduction and Preliminaries

Let L be a Lie algebra, then the *lower* and *upper central series* of L are defined as follows:

$$L = L^1 \supseteq L^2 \supseteq \cdots \supseteq L^{n+1} \supseteq \cdots,$$

and

$$(0) = Z_0(L) \subseteq Z_1(L) \subseteq = Z(L) \subseteq Z_2(L) \subseteq \cdots \subseteq Z_n(L) \subseteq \cdots,$$

respectively, where $L^{n+1} = [L, L^n]$ and $\frac{Z_n(L)}{Z_{n-1}(L)} = Z(\frac{L}{Z_{n-1}(L)})$.

The following definition is vital in our investigation and it is similar to the case for groups (see [1] or [2]).

Definition 1.1. Let L and K be two Lie algebras, $\alpha : \frac{L}{Z_n(L)} \longrightarrow \frac{K}{Z_n(K)}$ and $\beta : L^{n+1} \longrightarrow K^{n+1}$ be Lie homomorphisms such that the following diagram is commutative

$$\begin{array}{ccc}
\frac{L}{Z_n(L)} \times \cdots \times \frac{L}{Z_n(L)} & \stackrel{\varphi}{\longrightarrow} & L^{n+1} \\
\underbrace{\alpha \times \cdots \times \alpha}_{(n+1)-times} \downarrow & & \downarrow \beta \\
\underbrace{\frac{K}{Z_n(K)} \times \cdots \times \frac{K}{Z_n(K)}}_{(n+1)-times} & \stackrel{\psi}{\longrightarrow} & K^{n+1}
\end{array}$$

where $\varphi : (\bar{l_1}, \bar{l_2}, \cdots, \bar{l_{n+1}}) \mapsto [l_1, l_2, \cdots, l_{n+1}]$, for all $\bar{l_i} \in \frac{L}{Z_n(L)}$, $i = 1, \cdots, n+1$ and similarly for ψ. In fact, α and β are defined in such a way that they are compatible, i.e., for all $l_i \in L$, $\beta([l_1, l_2, \cdots, l_{n+1}]) = [k_1, k_2, \cdots, k_{n+1}]$, in which $k_i \in \alpha(l_i + Z_n(L)), 1 \leq i \leq n+1$.

The pair (α, β) is called *n-homoclinism* and if they are both isomorphisms, then (α, β) is called *n-isoclinism*. In this case, L and K are said to be *n-isoclinic*, which is denoted by $L \underset{n}{\sim} K$. If $n = 1$, then it will be the notion of isoclinism, which was first introduced by P.Hall [1] in 1940. The kernel and the image of (α, β) are defined as follows:

$$Ker(\alpha, \beta) = Ker\beta \;\text{ and }\; Im(\alpha, \beta) = I \subseteq K,$$

where $\alpha(\frac{L}{Z_n(L)}) = \frac{I}{Z_n(K)}$.

Now, the above definition gives the following

Theorem 1.2. Let (α, β) be an n-homoclinism of Lie algebras L into K, then

(*i*) $Ker(\alpha, \beta) \trianglelefteq L$;

(*ii*) $\frac{L}{Ker(\alpha,\beta)} \underset{n}{\sim} Im(\alpha, \beta)$.

2. Some properties of n-isoclinism of Lie algebras

In this section, some of the basic properties of n-isoclinism in Lie algebras are discussed.

The following lemma is useful in proving the next result and its proof is straightforward.

Lemma 2.1. Let L_1 and L_2 be two Lie algebras and $L = L_1 \oplus L_2$, then for all $n \geq 1$,

$$L^n = L_1^n \oplus L_2^n \ \text{ and } \ Z_n(L) = Z_n(L_1) \oplus Z_n(L_2).$$

Theorem 2.2. Let L be a Lie algebra and M be an abelian Lie algebra. Then for all $n > 1$

$$L \underset{n}{\sim} L \oplus M.$$

The following result is a criterion for two Lie algebras being n-isoclinic.

Theorem 2.3. Two Lie algebras L and K are n-isoclinic if and only if there exist ideals L_1 and K_1 of L and K contained in $Z_n(L)$ and $Z_n(K)$, respectively, and the isomorphisms $\alpha : L/L_1 \longrightarrow K/K_1$ and $\beta : L^{n+1} \longrightarrow K^{n+1}$ such that α induces β.

The following corollary is an immediate consequence of the above theorem.

Corollary 2.4. Let L and K be Lie algebras. If $L \underset{n}{\sim} K$, then $L \underset{m}{\sim} K$, for all $m \geq n$.

Theorem 2.5. Let (α, β) be the pair of n-isoclinism between two Lie algebras L_1 and L_2. Then, for all $n \geq 1$

(a) if M_1 is a subalgebra of L_1 containing $Z_n(L_1)$ such that $\alpha(M_1/Z_n(L_1)) = M_2/Z_n(L_2)$, then $M_1 \underset{n}{\sim} M_2$.

(b) if M_1 is an ideal of L_1 and $M_1 \subseteq L_1^{n+1}$, then $M_2 = \beta(M_1) \lhd L_2$ and $\frac{L_1}{M_1} \underset{n}{\sim} \frac{L_2}{M_2}$.

Theorem 2.6. Let K and M be subalgebra and ideal of a Lie algebra L, respectively. Then for all $n \geq 0$,

(a) $K \underset{n}{\sim} K + Z_n(L)$. In particular, if $L = K + Z_n(L)$ then $L \underset{n}{\sim} K$.

Conversely, if $\frac{L}{Z_n(L)}$ is finite dimension and $L \underset{n}{\sim} K$, then $L = K + Z_n(L)$.

(b) $\frac{L}{M} \underset{n}{\sim} \frac{L}{M \cap L^{n+1}}$. In particular, if $L \cap L^{n+1} = 0$, then $\frac{L}{M} \underset{n}{\sim} L$.

Conversely, if L^{n+1} is finite dimension and $L \underset{n}{\sim} \frac{L}{M}$, then $L^{n+1} \cap M = 0$.

Theorem 2.7. Let $f : L \longrightarrow K$ be a homomorphism of Lie algebras such that $f(Z_n(L)) \subseteq Z_n(K)$, for some $n \geq 1$. If

$$\begin{array}{rccc} \alpha : & \frac{L}{Z_n(L)} & \longrightarrow & \frac{K}{Z_n(K)} \\ & l + Z_n(L) & \mapsto & f(l) + Z_n(K)) \end{array}$$

and

$$\begin{array}{rccc} \beta : & L^{n+1} & \longrightarrow & K^{n+1} \\ & [l_1, \cdots, l_{n+1}] & \mapsto & [f(l_1), \cdots, f(l_{n+1})], \end{array}$$

for all $l, l_1, \cdots, l_{n+1} \in L$. Then the following statements hold:

(a) (α, β) is an n-homoclinism of L into K;

(b) $L/Kerf \underset{n}{\sim} L/Ker(\alpha, \beta)$;

(c) $Imf \underset{n}{\sim} Im(\alpha, \beta)$.

The following corollary gives a sufficient condition that two Lie algebras are n-isoclinic.

Corollary 2.8. If $f : L \longrightarrow K$ is an epimorphism of Lie algebras such that $Kerf \cap L^{n+1} = 0$, then $L \underset{n}{\sim} K$.

Proof . Since f is surjective, it follows that $f(Z_n(L)) \subseteq Z_n(K)$. Hence the pair (α, β) is defined in the above theorem is an n-homoclinism such that $Im(\alpha, \beta) = K$. Thus

$$L \cong \frac{L}{Kerf \cap L^{n+1}} \underset{n}{\sim} \frac{L}{Kerf} \cong K. \qquad \square$$

Theorem 2.9. Let K be a subalgebra of a Lie algebra L and $\frac{L}{Z_n(L)}$ satisfies the descending chain condition on subalgebras. Then the following are equivalent:

(a) $L = K + Z_n(L)$;

(b) $L \underset{n}{\sim} K$;

(c) $\frac{L}{Z_n(L)} \cong \frac{K}{Z_n(K)}$.

The following result gives some equivalence conditions on n-isoclinism of Lie algebras, which can be proved using Theorem 2.6(b).

Theorem 2.10. Let M be an ideal of a Lie algebra L. If L^{n+1} satisfies the ascending chain condition on ideals, for all $n \geq 1$, then the following conditions are equivalent:

(a) $M \cap L^{n+1} = 0$;

(b) $L \underset{n}{\sim} \frac{L}{M}$;

(c) $L^{n+1} \cong (\frac{L}{M})^{n+1}$.

the above theorems have the following corollary.

Corollary 2.11. Let K and M be a subalgebra and an ideal of a Lie algebra L, respectively.

(i) If N is an ideal of L and $M \cap L^{n+1} = 0$, then $L \underset{n}{\sim} \frac{L}{M\cap N}$.

(ii) If $M \cap L^{n+1} = 0$, then $\frac{K+M}{M} \underset{n}{\sim} K$.

(iii) If $L = K + Z_n(L)$, then $\frac{K+M}{M} \underset{n}{\sim} \frac{L}{M}$.

(iv) If $J \leq L$ and $L = K + Z_n(L)$, then $L \underset{n}{\sim} K + J$.

Proof. (i) Clearly, $(M \cap N) \cap L^{n+1} = 0$. Now, the result follows by Theorem 2.8 .

(ii) By the assumption, we have $(M \cap K) \cap K^{n+1} = 0$ and so $\frac{K+M}{M} \cong \frac{K}{M\cap K} \underset{n}{\sim} K$.

(iii) One observes that

$$\frac{L}{M} = \frac{K + Z_n(L)}{M} = \frac{K+M}{M} + \frac{Z_n(L) + M}{M}$$

and $\frac{Z_n(L)+M}{M} \leq Z_n(\frac{L}{M})$. Thus

$$\frac{L}{M} = \frac{K+M}{M} + Z_n(\frac{L}{M}).$$

Now, the result follows, using Theorem 2.7.

(iv) Clearly, $L = (K + L) + Z_n(L)$ and hence

$$L \underset{n}{\sim} K + J. \qquad \square$$

In the next section we study the concept of n-stem Lie algebras.

3. The structure of n-stem Lie algebras

In this section we introduce the concept of n-stem Lie algebras and similar to group theory case (see [1, 2]), it is shown that every family of n-isoclinic Lie algebras contains an n-stem Lie algebra of minimum dimension .

Definition 3.1. A Lie algebra L is said to be an n-stem Lie algebra, if $Z_n(L) \leq L^{n+1}$, for $n \geq 1$.

Theorem 3.2. Let $\mathcal{C}$ be a family of n-isoclinic Lie algebras, then

(a) $\mathcal{C}$ contains an n-stem Lie algebra;

(b) Suppose T is a finite dimensional Lie algebra in $\mathcal{C}$, then T is an n-stem Lie algebra if and only if

$$dimT = \min\{dimL \mid L \in \mathcal{C}\}.$$

The following lemma shortens the proof of our final result.

Lemma 3.3. Let L and M be two Lie algebras such that $L \underset{n}{\sim} M$ with the isomorphisms pair (α, β). Then, for all $x \in L^{n+1}$

(a) $\alpha(x + Z_n(L)) = \beta(x) + Z_n(M)$;

(b) $\beta([x, y]) = [\beta(x), m], \quad \forall y \in L,\ m \in \alpha(y + Z_n(L))$.

Theorem 3.4. If L and M are n-stem Lie algebras. Then $Z_n(L) \cong Z_n(M)$, for $n \geq 1$.

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