Some Properties of Autocommutator Groups

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Abstract

Let G be a group and $Aut(G)$ the group of automorphisms of G. For any element $g \in G$ and $\alpha \in Aut(G)$ the element $[g, \alpha] = g^{-1}g^{\alpha}$ is an *autocommutator* of g and α.

Also, the *autocommutator subgroup* of G is defined to be

$$K(G) =< [g, \alpha] = g^{-1}g^{\alpha} | g \in G, \alpha \in Aut(G) >,$$

which is a characteristic subgroup of G.

In this talk, we discuss some properties of this concept and its generalization.

1. introduction

Let $A = Aut(G)$ denote the group of automorphisms of a given group G. For any element $g \in G$ and $\alpha \in A$ the element $[g, \alpha] = g^{-1}g^{\alpha}$ is an *autocommutator* of g and α. Also, let $K(G) = \langle [g, \alpha] | g \in G, \alpha \in Aut(G) \rangle$ denote the *autocommutator subgroup* of G.

P. Hegarty ([2] and [3]), proved that for any given finite group G, the number $n(G)$ of the finite groups X such that $K(X) \cong G$ is finite. In fact he proves

Theorem (Hegarty (1997)). Given a finite group G, there are finitely many finite groups X satisfying $K(X) \cong G$.

Hegarty did not give any explicite bound for the number of solutions X of the equation $K(X) \cong G$; One notes that: For example, since the symmetric group S_3 is a complete group, it is easily seen that there are no groups X such that $K(X) \cong S_3$.

The above remark leads one to suspect that when the structure of a finite group G is very simple (i.e, G has very few automorphisms), the number of solutions of the equation $K(X) \cong G$ is quite small. This is in contrast with the equation $X' \cong G$; this equation, if it has a solution, has infinitely many solutions, for if X is a solution, then so is $X \times A$, where A is an arbitrary abelian group.

However, the following result is proved in [2] for cyclic groups.

Theorem (Deaconescu and Walls, 2007).

1) If $K(G) \cong Z$, then $G \cong Z$, $G \cong Z \times C_2$, or $G \cong D_\infty$.

2) Let G be a finite group such that $K(G) \cong C_p$. If $p = 2$, then $G \cong C_4$. If $p > 2$, then $G \cong C_p, C_p \times C_2, T$, or $T \times C_2$, where T is a partial holomorph of C_p.

Since there are finite groups G, as for instance $G = S_3$, such that $n(G) = 0$, a natural question is to determine those groups G satisfying $n(G) \geq 1$. So in [1], C. Chis, M. Chis and G. Silberberg prove the following result:

Theorem (C. Chis, M. Chis and G. Silberberg (2008)). Every finite abelian group is the autocommutator subgroup of some finite abelian group.

2. Higher autocommutators

Let $A = Aut(G)$ denote the group of automorphisms of a given group G. For any element $g \in G$ and $\alpha \in A$ the element $[g, \alpha] = g^{-1}g^{\alpha}$ is an *autocommutator* of g and α. We define the autocommutator of higher weight inductively as follows:

$$[g, \alpha_1, \alpha_2, ..., \alpha_i] = [[g, \alpha_1, \alpha_2, ..., \alpha_{i-1}], \alpha_i],$$

for all $\alpha_1, \alpha_2, ..., \alpha_i \in A$.

So the *autocommutator subgroup* of *weight* $i + 1$ is defined in the following way:

$$K_i(G) = [G, \underbrace{A, ..., A}_{i-times}] = \langle [g, \alpha_1, \alpha_2, ..., \alpha_i] \mid g \in G, \alpha_1, \alpha_2, ..., \alpha_i \in A \rangle.$$

Clearly $K_i(G)$ is a characteristic subgroup of G, for all $i \geq 1$. Therefore, one obtains a descending chain of autocommutator subgroups of G as follows:

$$G \supseteq K_1(G) \supseteq K_2(G) \supseteq ... \supseteq K_i(G) \supseteq ...,$$

which we may call it the *lower autocentral series* of G. The aim of this section is to prove the following main result.

Theorem 2.1. For any finite abelian group G and every natural number $n \in N$, there exists a finite abelian group H such that

$$G \cong K_n(H).$$

Using the above notation, we have the following

Lemma 2.2. If Z_m is a finite cyclic group, then for any natural number n,

$$K_n(Z_m) = Z_m^{2^n}.$$

Lemma 2.3. Let G be a finite abelian group of odd order m and Z_2 the cyclic group of order 2, then $K_n(G)$ and $K_n(G \times Z_2)$ are both isomorphic with G, for all natural number n.

Theorem 2.4. For all natural numbers $m \geq n_1 \geq ... \geq n_r$ and $n \geq 2$, then

$$K_n(Z_{2^m} \times Z_{2^{n_1}} \times ... \times Z_{2^{n_r}}) = Z_{2^{m-n}} \times Z_{2^{n_1-(n-1)}} \times ... \times Z_{2^{n_r-(n-1)}}.$$

Proof of Theorem 2.1

Let G be a finite abelian group, which can be written as a product of its Sylow subgroups. Now, if $|G|$ is an odd number then by Lemma 2.3,

$$G = K_n(G).$$

Assume 2 divides $|G|$ and A is the Sylow 2-subgroup of G, then $G = A \times P_1 \times ... \times P_s$, where $P_i^{'}s$ are Sylow p_i-subgroups of $G, (1 \leq i \leq r)$. By Lemma 2.1,

$$K_n(G) = K_n(A) \times P_1 \times ... \times P_s.$$

As A is abelian 2-group, we may write A as a direct product of cyclic groups of orders some powers of 2, as follows:

$$A \cong Z_{2^m} \times Z_{2^{n_1}} \times ... \times Z_{2^{n_r}},$$

where $m \geq n_1 \geq ... \geq n_r$.

Now, we choose the abelian group

$$H = Z_{2^{m+n}} \times Z_{2^{n_1+n-1}} \times ... \times Z_{2^{n_r+(n-1)}} \times P_1 \times ... \times P_s.$$

It can be easily seen that

$$K_n(H) = G,$$

and hence the claim is proved. $\square$

3. Autonilpotent groups

The following definition is vital in our investigations.

Definition 3.1.

We call the set of elements

$$L(G) = \{g \in G \mid [g, \alpha] = 1 \ \ or \ \ g^{\alpha} = g, \ \forall \alpha \in A\}$$

the *autocentre* of G.

Clearly, it is a characteristic subgroup of G (see [1] for more information) and if $A = \mathrm{Inn}(G)$ then $L(G) = Z(G)$ is the centre of G.

Now, we define the *upper autocentral series* of G in following way:

$$\langle 1 \rangle = L_0(G) \subseteq L_1(G) = L(G) \subseteq L_2(G) \subseteq ... \subseteq L_n(G) \subseteq ...,$$

where $\frac{L_n(G)}{L_{n-1}(G)} = L(\frac{G}{L_{n-1}(G)})$, or equivalently $L_n(G) = \pi_G^{-1}(L(\frac{G}{L_{n-1}(G)}))$, for all $n \geq 2$, in which $\pi_G : G \longrightarrow \frac{G}{L_{n-1}(G)}$ is a homomorphism.

In particular, if we take the group of inner automorphisms we obtain the usual upper central series of G. A group G is said to be *autonilpotent group* of class at most n if $L_n(G) = G$, for some natural number $n \in N$.

The main purpose of this section is to determine all finite abelian groups, which are autonilpotent.

The following lemma follows easily from the definition.

Lemma 3.2. Let G be a group and $x \in L_n(G)$, for some $n \geq 1$, then for all $\alpha_1, ..., \alpha_n \in Aut(G)$,

$$[x, \alpha_1, ..., \alpha_n] = 1.$$

Corollary 3.3. If G is an autonilpotent group of class n, then $K_n(G) = \langle 1 \rangle$.

Remark. For any group G and each natural number n,

$$K_n(G) \geq \gamma_n(G),$$

and

$$L_n(G) \leq Z_n(G).$$

Note that the above inequalities will be attained, when the group G is taken to be the symmetric group S_3, since $Aut(S_3) = \mathrm{Inn}(S_3)$.

One observes that autonilpotent groups are nilpotent, but the converse is not true in general.

Example 3.4 One can easily check that
$L(Z_2) = Z_2; L(Z_3) = \langle 1 \rangle; L_2(Z_4) = Z_4; L(Z_6) = \{e, x^3\}$ and $L_2(Z_6) = L(Z_6)$.

Hence the cyclic groups of orders 2 and 4 are autonilpotent and the ones of orders 3 and 6 are *not*, while they are nilpotent in the usual sense.

It is known that the symmetric group S_3 is not nilpotent and it is easily checked that $L(S_3) = \langle 1 \rangle$, hence it is *not* autonilpotent as well.

The following property for autonilpotent groups is similar to the one in the usual nilpotent groups.

Proposition 3.5. If G is a non-trivial autonilpotent group, then

$$L(G) \neq \langle 1 \rangle.$$

4. Some more properties of autonilpotent groups

In this section it is shown that some of the known results of nilpotent groups can be carried over to autonilpotent groups.

In the view of Example 3.4 in the previous section, we show that all cyclic groups of order $2^n, n > 1$, are autonilpotent, while it is not so for arbitrary cyclic groups.

Remark. The cyclic group Z_p, of odd prime order p is *not* autonilpotent, while it is nilpotent, since $Aut(Z_p) = U_{p-1}$ is a cyclic group of order $p - 1$ and so it can not fix any element of Z_p.

It will be shown that this is held for all cyclic groups of order p^n, when $p \neq 2$ (see Theorem 4.3, below). On the other hand, if $p = 2$ the following result shows that the cyclic group of order 2^n is autonilpotent.

Theorem 4.1. The cyclic group, Z_{2^n}, of order 2^n $(n > 1)$, is an autonilpotent group.

Proof. Let $Z_{2^n} = \langle x \mid x^{2^n} = 1 \rangle$ be the cyclic group of order 2^n $(n > 1)$. Clearly, if $r = 2t + 1$ is an odd number then the map $\alpha : x \mapsto x^r$ is an automorphism. So

$$r2^{n-1} = (2t + 1)2^{n-1} \equiv 2^{n-1} \pmod{2^n}.$$

Hence $(x^{2^{n-1}})^{\alpha} = x^{r2^{n-1}} = x^{2^{n-1}}$. Now, if for each $\alpha \in Aut(Z_{2^n})$ and $s \in N$,

$$(x^s)^{\alpha} = x^s,$$

then $x^{rs} = x^s$ and so $x^{s(r-1)} = 1$, which implies that $2^n \mid s(r-1)$, i.e. $s = 2^{n-1}$. Thus $L(Z_{2^n}) = \{e, x^{2^{n-1}}\}$ and hence $L(\frac{Z_{2^n}}{L(Z_{2^n})})$ is a cyclic group of order 2^{n-1}. Continuing in this way we obtain $L_n(Z_{2^n}) = Z_{2^n}$, after n-steps, which proves the claim. $\square$

Theorem 4.2. (i) Let C_2 and Z_{2^n} be the cyclic groups of orders 2 and $2^n, (n > 1)$, respectively. Then the direct product $C_2 \times Z_{2^n}$ is not autonilpotent group.

(ii) The direct product $C_{2^m} \times Z_{2^n}$ can not be autonilpotent group, for all natural number m, n.

Proof. (i) Let $Z_{2^n} = \langle x | \ x^{2^n} = 1 \rangle$ and $C_2 = \langle y | \ y^2 = 1 \rangle$ be the cyclic groups of orders 2^n and 2, respectively. Then clearly the action of any automorphism $\alpha \in Aut(C_2 \times Z_{2^n})$ on the generators x and y is as follows:

$$\alpha(x) = x^i, (i, 2^n) = 1, \quad \alpha(y) = y^j x^{2^{n-1}}, j = 0, 1.$$

By the definition of the autocentre of G, for all $y^r x^s \in L(C_2 \times Z_{2^n})$, we have $\alpha(y^r x^s) = y^r x^s$ or $\alpha(y^r)\alpha(x^s) = y^r x^s$, which implies $y^{rj} x^{r2^{n-1}}.x^{si} = y^r x^s$. Thus $y^{r(j-1)} x^{r2^{n-1}+s(i-1)} = 1$. Hence $2|r(j-1)$ and $2^n | r2^{n-1} + s(i-1)$. Now, since $j = 0, 1$ it follows that $r = 2t, t \geq 1$ and $2^n | s(i-1)$, for all i, where $(i, 2^n) = 1$. So it must be true for $i = 3$, which implies that $s = 2^{n-1}$ and so $y^r x^s = 2^{n-1}$. This follows that

$$L(C_2 \times Z_{2^n}) = \langle x^{2^{n-1}} \rangle \cong Z_2.$$

Hence

$$L_n(C_2 \times Z_{2^n}) = L_{n-1}(C_2 \times Z_{2^n}) = Z_{2^{n-1}}.$$

Therefore, $C_2 \times Z_{2^n}$ can not be autonilpotent group.

(ii) Using a similar argument as in part (i), one may prove this part.□

The following theorem gives a complete characterization of cyclic groups.

Theorem 4.3. The cyclic group, Z_{p^n}, of order p^n, is not autonilpotent, for each odd prime p and $n \geq 1$.

Proof. Let $Z_{p^n} = \langle a \mid a^{p^n} = 1 \rangle$ be the cyclic group of order p^n, where $p \neq 2$ and $n \geq 1$. Then $\phi : a \longmapsto a^2$ is an automorphism of Z_{p^n}. If there exists $i < p^n$ such that $\phi(a^i) = a^i$, then $a^{2i} = a^i$ and so $a^i = 1$, which contradicts the order of a. Thus $L(G) = \langle 1 \rangle$ and hence by Proposition 1.5, Z_{p^n} can not be an autonilpotent group. □

Example 4.4 Consider $D_8 = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1} \rangle$, the dihedral group of order 8. Clearly, the group of automorphisms of D_8 is of order 8 and one may check that $L(D_8) = \{e, a^2\} \cong Z_2$ and

$$\frac{L_2(D_8)}{L(D_8)} = L(\frac{D_8}{L(D_8)}) = L(Z_2 \times Z_2) = \langle 1 \rangle.$$

Hence $L_2(D_8) = L(D_8)$, which implies that D_8 is not an autonilpotent group, while it is nilpotent.

The following result gives the basic step in proving our main goal which says that; the abelian groups, which are autonilpotent are the only cyclic groups of order 2^n, for $n \geq 1$.

Theorem 4.5. If the group $G = H \times K$ is the direct product of its characteristic subgroups H and K, then for all $n \geq 1$,

$$L_n(H \times K) = L_n(H) \times L_n(K).$$

The following corollaries are immediate consequences of the above theorem.

Corollary 4.6. If $(|H|, |K|) = 1$, then the above theorem is also true.

Corollary 4.7. If $G = H \times K$, is the direct product of its characteristic subgroups such that H or K is not autonilpotent, then so is not G.

Our final result classify all finite abelian groups which are autonilpotent.

Theorem 4.8. A finite abelian group is autonilpotent if and only if it is a cyclic 2-group.

Proof. The necessary condition follows from Theorem 2.1. Now, for the reverse conclusion, we assume that G is not a cyclic 2-group. So it is either abelian 2-group or G has a direct summand Z_{p^t}, where p is an odd prime number and $t \geq 1$. In the first case, Theorem 2.2 implies that the group G is not autonilpotent in the second case by Theorem 4.3, Z_{p^t} can not be autonilpotent. Thus, Corollary 4.7 gives the result. $\square$

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