GROUPS SATISFYING A SYMMETRIC ENGEL WORD

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ABSTRACT. In this article it is shown that a finite group satisfying $[y,_n x] = [x,_n y]$ $(n > 1)$ is nilpotent and that if G is a group satisfying $[y,x] = [x,y]$, then

$$[\gamma_3(G), \gamma_2(G)] = [\gamma_2(G), \gamma_2(G), G] = 1.$$

Also, we investigate groups satisfying both $[y,x] = [x,y]$ and $[y,_n x] = [x,_n y]$ for small n.

Our results can be applied to obtain special commutators, which can be expressed as the product of commutators squares.

INTRODUCTION

Let G be a finite group. A word $w = w(x,y)$ is called *symmetric* on the group G if $w(g_1, g_2) = w(g_2, g_1)$, for all $g_1, g_2 \in G$. Now, let $E_n = E_n(x,y) = [y,_n x]$ be the nth Engel word. Then G is said to be an E_n-symmetric group if E_n is symmetric on G. If G is finite and $E_n \equiv 1$, then it is known that G is nilpotent. In this paper we shall generalized this result by showing that G is still nilpotent if E_n $(n \geq 2)$ is symmetric on G.

If P is an elementary abelian 2-group and ϕ is a fixed-point-free automorphism of P of odd prime order p then semi-direct product $G = P \rtimes \langle \phi \rangle$ is clearly a finite E_1-symmetric group, which is not nilpotent. So, E_1-symmetric groups are not necessarily nilpotent. We will show that an E_1-symmetric group is near metabelian and in finite case it is an extension of a 2-group by an abelian group of odd order. We will also present some more results concerning groups, which are both E_1- and E_n-symmetric for small n and we give conditions, on which some commutators of weight > 1 can be expressed as the product of commutators squares.

1. E_n-SYMMETRIC GROUPS, $n \geq 2$

It is well-known that a finite Engel group is nilponent (see [4, Theorem 12.3.4]). Now, we generalize this result by showing that a finite group satisfying a symmetric n-Engel word $(n \geq 2)$ is also nilpotent.

Theorem 1.1. *If G is a finite E_n-symmetric group $(n \geq 2)$, then G is nilpotent.*

Proof. First, suppose that G is solvable. Clearly, $[y,x] \in G^{(1)}$ and if $[y,_{1+kn} x] \in G^{(k+1)}$ then

$$[y,_{1+(k+1)n} x] = [y,_{1+kn} x,_n x] = [x,_n [y,_{1+kn} x]] \in G^{(k+2)}.$$

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Hence, we reach to $[y,_{1+(m-1)n}\, x] = 1$ by choosing m to be the solvability length of G that is G is an Engel group. Using [4, Theorem 12.3.4] we conclude that G is nilpotent. Now, suppose that G is a finite E_n-symmetric group and the result holds for all groups of order less than $|G|$. Since, the proper subgroups of G inherit the same property as G does, each of which should be nilpotent. Hence, by [4, Theorem 9.1.9], G is solvable and consequently G is nilpotent. □

Theorem 1.1 can be generalized in the following form.

Corollary 1.2. *Let G be a finite group. If for each $x, y \in G$ there exist integers $m_{x,y}, n_{x,y} > 1$ such that $[y,_{m_{x,y}}\, x] = [x,_{n_{x,y}}\, y]$, then G is nilpotent.*

2. E_1-symmetric groups

E_1-symmetric groups are different from E_n-symmetric groups ($n \geq 2$) as they are not nilpotent in general. We show that every E_1-symmetric group is solvable of length at most 3. But, we have no proof that whether they are metabelian or not.

Example. Let F be a field of characteristic 2 and let $G = U(n, F)$ be the Unitriangular group of matrices of dimension $n \leq 4$ over F. Then G is an E_1-symmetric group.

To prove our main results we first need some elementary properties of E_1-symmetric groups.

Lemma 2.1. *Let G be an E_1-symmetric group. Then*
i) $[[x,_m\, y], [x,_n\, y]] = 1$;
ii) $[[x, y], [x, z]] = 1$;
iii) $[[x, y_1, \ldots, y_m], [x, z_1, \ldots, z_n]] = 1$;
iv) $[[x_1, x_2], [x_3, x_4]] = [[x_1^\pi, x_2^\pi], [x_3^\pi, x_4^\pi]]$;
v) $[[x, y], [z, w]] = [x, y, z, w][x, y, w, z]$,
where $x, y, z, w, x_1, \ldots, x_4, y_1, \ldots, y_m, z_1, \ldots, z_n \in G$, $\pi \in S_4$ and m, n are natural numbers.

Proof. i) We proceed by induction on n to prove that $[[x, y], [x,_n\, y]] = 1$ and $[x, y^n] = [x, y]^{\epsilon_1}[x,_2\, y]^{\epsilon_2} \cdots [x,_n\, y]^{\epsilon_n}$ for each $x, y \in G$, $n \geq 1$ and for some $\epsilon_i \in \{0, 1\}$ depending on n and $\epsilon_n = 1$, by which part (i) would follow. Clearly, the result holds, when $n = 1$. Now, we assume that $n \geq 1$ and the the result holds for $1, \ldots, n$. Thus, we have

$$\begin{aligned} [x, y^{n+1}] &= [x, y^n][x, y][x, y, y^n] \\ &= [x, y]^{\epsilon_1} \cdots [x,_n\, y]^{\epsilon_n}[x, y][x,_2\, y]^{\epsilon_1}[x,_3\, y]^{\epsilon_2} \cdots [x,_{n+1}\, y]^{\epsilon_n} \\ &= [x, y]^{\epsilon_1+1}[x,_2\, y]^{\epsilon_1+\epsilon_2} \cdots [x,_n\, y]^{\epsilon_{n-1}+\epsilon_n}[x,_{n+1}\, y]^{\epsilon_n} \end{aligned}$$

for some $\epsilon_1, \ldots, \epsilon_{n-1} \in \{0, 1\}$ and $\epsilon_n = 1$. Let $\epsilon'_1, \ldots, \epsilon'_{n+1} \in \{0, 1\}$ be equal to $\epsilon_1 + 1, \epsilon_1 + \epsilon_2, \ldots, \epsilon_{n-1} + \epsilon_n, \epsilon_n$ modulo 2, respectively. Then, we get $[x, y^{n+1}] = [x, y]^{\epsilon'_1} \cdots [x,_{n+1}\, y]^{\epsilon'_{n+1}}$, where $\epsilon'_1, \ldots, \epsilon'_n \in \{0, 1\}$ and $\epsilon'_{n+1} = 1$. On the other hand, we have

$$\begin{aligned}[x,y^{n+1}] &= [y^{n+1},x]^{-1}\\ &= ([y,x][y,x,y^n][y^n,x])^{-1}\\ &= ([x,y][x,y,y^n][x,y^n])^{-1}\\ &= ([x,y][x,_2 y]^{\epsilon_1}\cdots[x,_{n+1} y]^{\epsilon_n}[x,y]^{\epsilon_1}\cdots[x,_n y]^{\epsilon_n})^{-1}\\ &= [x,_n y]^{\epsilon_n}\cdots[x,y]^{\epsilon_1}[x,_{n+1} y]^{\epsilon_n}\cdots[x,_2 y]^{\epsilon_1}[x,y]\\ &= [x,_2 y]^{\epsilon_1+\epsilon_2}\cdots[x,_n y]^{\epsilon_{n-1}+\epsilon_n}[x,y]^{\epsilon_1}[x,_{n+1} y]^{\epsilon_n}[x,y].\end{aligned}$$

By comparing these two identities we obtain that $[x,_{n+1} y][x,y]=[x,y][x,_{n+1} y]$, as was claimed.

ii) Let $x,y,z\in G$. If $g\in G$ then $[x,g]=[g,x]$ that is $g^{-1}g^x=g^{-x}g$. Hence, $g^x=gg^{-x}g$. As $(yz)^x=y^xz^x$ we have

$$yz(yz)^{-x}yz=yy^{-x}yzz^{-x}z,$$

which implies that $[y^{-x}y,zz^{-x}]=1$. Replacing z by z^{-1} in the last identity and using the fact that $[z,x]=[x,z]$, we observe that $[[x,y],[x,z]]=[y^{-x}y,z^{-1}z^x]=1$, as required.

iii) To prove this part, we proceed by induction on (m,n). The case that $(m,n)=(1,1)$ follows by (ii). Now, assume that the result holds for (m,n). Then, expanding $[[x,y_1,\dots,y_{m-1},y_my_{m+1}],[x,z_1,\dots,z_n]]=1$ we obtain the result for $(m+1,n)$. Similarly, we can get the result for $(m,n+1)$, by which we conclude the result for all (m,n).

iv) The result would follow easily by expanding the identity $[[xy,z],[xy,w]]=1$ in conjunction with (iii).

v) Let $x,y,z,w\in G$. Then, we have

$$[x,y,zw]=[x,y,w][x,y,z][x,y,z,w]$$

and by applying (iii),

$$\begin{aligned}[x,y,zw] &= [x,y,wz[z,w]]\\ &= [[x,y],[z,w]][x,y,wz]^{[z,w]}\\ &= [[x,y],[z,w]][x,y,z]^{[z,w]}[x,y,w]^{[z,w]}[x,y,w,z]^{[z,w]}\\ &= [[x,y],[z,w]][x,y,z][x,y,w][x,y,w,z].\end{aligned}$$

From these two identities and applying (iii) once more we obtain the result. □

Theorem 2.2. *Let G be an E_1-symmetric group. Then*
i) $[\gamma_3(G),\gamma_2(G)]=1$*;*
ii) $[\gamma_2(G),\gamma_2(G),G]=1$.

Proof. i) It is well-known in the literature that any commutator is a product of squares. In fact, for $x,y\in G$ we have $[x,y]=x^{-1}y^{-1}xy=x^{-2}(xy^{-1})^2y^2$. Now, if $a,b,c,d,e\in G$, then

$$\begin{aligned}[a,b,c] &= [c,[a,b]]=[c,u^2v^2w^2]\\ &= [c,w^2][c,v^2]^{w^2}[c,u^2]^{v^2w^2}=[x_1,y_1^2][x_2,y_2^2][x_3,y_3^2]\end{aligned}$$

for some $u,v,w,x_1,y_1,x_2,y_2,x_3,y_3\in G$. By Lemma 2.1(iii,iv), we have

$$[[x_i,y_i^2],[d,e]]=[[x_i,y_i,y_i],[d,e]]=[[d,y_i],[x_i,y_i,e]]=1.$$

Therefore,

$$[[a,b,c],[d,e]] = [[x_1,y_1^2][x_2,y_2^2][x_3,y_3^2],[d,e]] = 1.$$

ii) Let $a,b,c,d,e \in G$. Then, by Lemma 2.1(v)

$$[[a,b],[c,d,e]] = [a,b,[c,d],e][a,b,e,[c,d]] = [[a,b],[c,d],e][[a,b,e],[c,d]].$$

Now, by applying part (i), we obtain $[[a,b],[c,d],e] = 1$, as required. □

It is investigated by several authors that, when a commutator (or an expression involving commutators) can be expressed as the product of special elements of the group, say squares, cubes etc? For example, it is proved that any commutator $[y,x]$ is the product of squares, $[y,x,x]$ is the product of cubes and the fifth Engel word $[y,x,x,x,x,x]$ is the product of forth powers (see [1, 2]).

Using Theorem 2.2, we observe that in an arbitrary group G the commutators of the form $[[a,b,c],[d,e]]$ and $[[a,b],[c,d],e]$ can be expressed as the product of commutators squares. Also, one should be able to prove that if F/F'^2 is centerless, where F is the free group of rank 4, then $[[a,b],[c,d]]$ can be expressed as the product of commutators squares.

The structure of finite E_1-symmetric groups can be describe in an alternative way as follows.

Theorem 2.3. *If G is a finite E_1-symmetric group, then G is a semidirect product of a normal Sylow 2-subgroup by an abelian subgroup of odd order.*

Proof. Let $x \in G$ be a 2-element of order 2^n. Then, $[y,_{2^n} x] = [y,x^{2^n}] = 1$ for each $y \in G$ and consequently x is a right Engel element. By [3] the set of all right Engel elements of G coincides with the Fitting subgroup $F(G)$ of G. Thus, $F(G)$ possesses all Sylow 2-subgroups of G. Let P be a Sylow 2-subgroup of G (hence of $F(G)$). As $F(G)$ is a characteristic nilpotent subgroup of G its Sylow 2-subgroup P is normal in G and hence by Schur-Zassenhaus theorem [4, Theorem 9.1.2], P has a complement H in G. Since H is of odd order it is abelian and the proof is complete. □

3. E_1- AND E_n-SYMMETRIC GROUPS, $n \geq 2$

In this section, we investigate groups satisfying both E_1- and E_n-symmetric properties for small n. We will show that in an E_1-symmetric group both E_2- and E_3-symmetric properties are equivalent to the 2- and 3-Engel properties, respectively.

Lemma 3.1. *If G is an E_1- and E_n-symmetric group ($n \geq 2$), then G is an $(n+1)$-Engel group.*

Proof. Let $x,y \in G$. Then, by Lemma 2.1(i)

$$[y,_{n+1} x] = [[y,x],_n x] = [x,_n [y,x]] = [[y,x,x],[y,x],_{n-2} [y,x]] = 1.$$

□

Lemma 3.2. *If G is an E_1-symmetric group, then $[y,x,x,y] = [x,y,y,x]$, for all $x,y,\in G$.*

Proof. If $x, y \in G$, then

$$[y, x, x, y] = [y, x^2, y] = [x^2, y, y] = [x^2, y^2]$$

and

$$[x, y, y, x] = [x, y^2, x] = [y^2, x, x] = [y^2, x^2],$$

from which the result follows. □

Theorem 3.3. *If G is both E_1- and E_2-symmetric group, then G is nilponent of class at most* 2.

Proof. Since $[y, x, x] = [x, y, y]$ holds for all $x, y \in G$, then by expanding $[xy, x, x] = [x, xy, xy]$ we obtain $[y, x, x] = 1$, that is G is a 2-Engel group. Now, let $x, y, z \in G$. Then

$$\begin{aligned} [x, y, z] &= [z, [x, y]] = [z, u^2v^2w^2] \\ &= [z, w^2][z, v^2]^{w^2}[z, u^2]^{v^2w^2} = [z, w, w][z, v, v]^{w^2}[z, u, u]^{v^2w^2} = 1, \end{aligned}$$

for some $u, v, w \in G$. Hence, G is nilpotent of class at most 2. □

Theorem 3.3 asserts that in an arbitrary group G and elements $x, y, z \in G$, there always exist elements x_i, y_i and z_i, w_i such that

$$[x, y, z] = \prod [x_i, y_i]^2 \prod [z_i, w_i, w_i][w_i, z_i, z_i]^{-1}.$$

Thus, the commutators $[x, y, z]$ can be expressed as the product of commutators squares if and only if the elements $[y, x, x][x, y, y]^{-1}$ have the same property.

Theorem 3.4. *If G is both E_1- and E_3-symmetric group, then G is a* 3*-Engel group.*

Proof. For all x and $y \in G$, by expanding the identity $[y, yx, yx, yx] = [yx, y, y, y]$ and utilizing Lemmas 3.1 and 3.2, we obtain

$$[y, x, x, x] = [y, x, y, x, y][y, x, y, x, y, x] = [y, x, y, x, y]^x,$$

which implies that

$$[y, x, x, x] = [y, x, y, x, y].$$

By replacing y by y^2 in the last identity we get

$$\begin{aligned} [x, y, x, y, x] &= [y, x, x, y, x] = [x, y, y, x, x] = [x, y^2, x, x] = [y^2, x, x, x] \\ &= [y^2, x, y^2, x, y^2] = [x, y^2, y^2, x, y^2] = [x, y, y, y, y, x, y^2] = 1, \end{aligned}$$

from which we also get $[y, x, y, x, y] = 1$ and consequently $[y, x, x, x] = 1$, as required. □

Theorem 3.5. *If G is both E_1- and E_3-symmetric group, then $G'' = \langle 1 \rangle$.*

Proof. Expanding the identity $[z, xy, xy, xy] = 1$ in conjunction with the fact that $[a, b, c, d] = [a, b, d, c]$ by Lemma 2.1(ii,v), when $a, b, c, d \in \{x, y, z\}$ we would obtain $[z, x, y, y] = [z, y, x, x]$. Moreover

$$[z, y, y, x, x] = [z, y^2, x, x] = [z, x, y^2, y^2] = [z, x, y, y, y, y] = 1,$$

from which we deduce that $[w, z, y, x, x] = 1$, for all $x, y, z, w \in G$.

Now, expanding $[w, z, xy, xy] = [w, xy, z, z]$ in conjunction with the previous identities we reach to $[w, z, x, y] = [w, z, y, x]$. Therefore $G'' = \langle 1 \rangle$. □

From Theorem 3.5, we conclude that each commutator $[[x, y], [z, w]]$ can be expressed as

$$[[x, y], [z, w]] = \prod [x_i, y_i]^2 \prod [z_i, w_i, w_i, w_i][w_i, z_i, z_i, z_i]^{-1}.$$

Hence, in an arbitrary group G, the commutators $[[x, y], [z, w]]$ can be expressed as the product of commutators squares if the elements $[y, x, x, x][x, y, y, y]^{-1}$ have the same property.

References

[1] A. Abdollahi, Powers of a products of a commutators as products of squares, Int. J. Math. Math. Sci., **7** (2004), 373-375.

[2] G. Havas, Commutators in groups expressed as products of powers, Comm. Algebra, **9**(2) (1981), 115-129.

[3] T.A. Peng, Normal p-subgroups of finite groups, J. London Math. Soc., **2**(8) (1971), 301-306.

[4] D.J. Robinson, A Course in the Thoery of Groups, Springer-Verlag, Berlin, 1982.

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