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SOME NOTES ON POLYNILPOTENT MULTIPLIERS OF p-GROUPS

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ABSTRACT. In this talk, we show that if $G = \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} \ldots \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$ is the nth nilpotent product of some cyclic p-groups, where $c_1 \geq n$, $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_t$ and $(q,p) = 1$ for all prime q less than or equal to n, then $|\mathcal{N}_{c_1,c_2,\ldots,c_s}M(G)| = p^{d_m}$ if and only if $G = \underbrace{\mathbf{Z}_p \overset{n}{*} \mathbf{Z}_p \overset{n}{*} \ldots \overset{n}{*} \mathbf{Z}_p}_{m-copies}$, where $m = \sum_{i=1}^{t} \alpha_i$ and $d_m = \chi_{c_s+1} \ldots (\chi_{c_2+1}(\sum_{j=1}^{n} \chi_{c_1+j}(m) \ldots)$.

1. INTRODUCTION

Let G be any group with a free presentation $G \cong F/R$. Then the Baer invariant of G with respect to the variety of groups $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be $\mathcal{V}M(G) = (R \cap V(F))/[RV^*F]$, where V is the set of words of the variety $\mathcal{V}$, $V(F)$ is the verbal subgroup of F and

$$[RV^*F] = \langle v(f_1,\ldots,f_{i-1},f_ir,f_{i+1},\ldots,f_n)v(f_1,\ldots,f_i,\ldots,f_n)^{-1} |$$
$$r \in R, f_i \in F, v \in V, 1 \leq i \leq n, n \in N \rangle.$$

One may check that $\mathcal{V}M(G)$ is abelian and independent of the choice of the free presentation of G. In particular, if $\mathcal{V}$ is the variety of abelian groups, $\mathcal{A}$, then the Baer invariant of the group G will be $(R \cap F')/[R,F]$, which is isomorphic to the well-known notion the Schur multiplier of G, denoted by $M(G)$. If $\mathcal{V}$ is the variety of polynilpotent groups of class row $(c_1,\ldots,c_t)$, $\mathcal{N}_{c_1,c_2,\ldots,c_t}$, then the Baer invariant of a group G with respect to this variety, which is called a polynilpotent multiplier of G, is as follows:

$$\mathcal{N}_{c_1,c_2,\ldots,c_t}M(G) = \frac{R \cap \gamma_{c_t+1} \circ \ldots \circ \gamma_{c_1+1}(F)}{[R, {}_{c_1}F, {}_{c_2}\gamma_{c_1+1}(F), \ldots, {}_{c_t}\gamma_{c_{t-1}+1} \circ \ldots \circ \gamma_{c_1+1}(F)]},$$

where $\gamma_{c_t+1} \circ \ldots \circ \gamma_{c_1+1}(F) = \gamma_{c_t+1}(\gamma_{c_{t-1}+1}(\ldots(\gamma_{c_1+1}(F))\ldots))$ are the term of iterated lower central series of F. In particular, if $t = 1$ and $c_1 = c$ then the Bear invariant of

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G with respect to the variety $\mathcal{N}_c$, which is called the c-nilpotent multiplier of G, is $\mathcal{N}_cM(G) = (R \cap \gamma_{c+1}(F))/[R, {}_cF]$.

Historically, in 1956, J.A. Green [2] showed that the order of the Schur multiplier of a finite p-group of order p^n is bounded by $p^{\frac{n(n-1)}{2}}$. In 1991, Ya.G. Berkovich [1] showed that a finite p-group of order p^n is an elementary ablian p-group if and only if the order of $M(G)$ is $p^{n(n-1)/2}$. In 1981, M.R.R. Moghaddam (see theorem 3.2 [5]) presented a bound for the polynilpotent multiplier of a finite p-group. He showed that if $\mathcal{V}$ is the variety of polynilpotent groups of a given class row and G is a finite d-generator group of order p^n, then $|\mathcal{V}M(G)||V(G)| \le |\mathcal{V}M(\mathbf{Z}_p^{(n)})|$, where $\mathbf{Z}_n^{(m)}$ denotes the direct sum of m copies of $\mathbf{Z}_n$. In 2005, the second author and M.A. Sanati [5] extended the result of Ya.G. Berkovich to the c-nilpotent multiplier of a finite p-group. They showed that for an abelian p-group G, $|\mathcal{N}_cM(G)| = p^{\chi_{c+1}(n)}$ if and only if G is an elementary ablian p-group, where $\chi_{c+1}(n)$ is the number of basic commutators of weight $c+1$ on n letters.

We show that if $\mathcal{V}$ is the variety of polynilpotent groups of class row $(c_1, c_2, ..., c_s)$, $\mathcal{N}_{c_1,c_2,...,c_s}$, and $G = \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$ is the nth nilpotent product of some cyclic p-groups, where $c_1 \ge n$, $\alpha_1 \ge \alpha_2 \ge ... \ge \alpha_t$ and $(q, p) = 1$ for all prime q less than or equal to n, then $|\mathcal{N}_{c_1,c_2,...,c_s}M(G)| = p^{d_m}$ if and only if $G = \underbrace{\mathbf{Z}_p \overset{n}{*} \mathbf{Z}_p \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_p}_{m-copies}$, where $m = \sum_{i=1}^{t} \alpha_i$ and $d_m = \chi_{c_s+1} ... (\chi_{c_2+1}(\sum_{j=1}^{n} \chi_{c_1+j}(m)...)$.

Definition 1.1. Let $\{G_i\}_{i\in I}$ be a family of arbitrary groups. The nth nilpotent product of the family $\{G_i\}_{i\in I}$ is defined as follows:

$$\overset{n}{\underset{i\in I}{\prod^{*}}} G_i = \frac{\prod_{i\in I}^{*} G_i}{\gamma_{n+1}(\prod_{i\in I}^{*} G_i) \cap [G_i]_{i\in I}^{*}},$$

where $\prod_{i\in I}^{*} G_i$ is the free product of the family $\{G_i\}_{i\in I}$, and

$$[G_i]_{i\in I}^{*} = \langle [G_i, G_j] | i, j \in I, i \ne j \rangle^{\prod_{i\in I}^{*} G_i}$$

is the cartesian subgroup of the free product $\prod_{i\in I}^{*} G_i$ which is the kernel of the natural homomorphism from $\prod_{i\in I}^{*} G_i$ to the direct product $\prod_{i\in I}^{\times} G_i$. If $\{G_i\}_{i\in I}$ is a family of cyclic groups, then $\gamma_{n+1}(\prod_{i\in I}^{*} G_i) \subseteq [G_i]^{*}$ and hence $\overset{n}{\prod_{i\in I}^{*}} G_i = \prod_{i\in I}^{*} G_i / \gamma_{n+1}(\prod_{i\in I}^{*} G_i)$

Theorem 1.2. [5] *Let G be a finite d-generator p-group of order p^n, then*

$$p^{\chi_{c+1}(d)} \le |\mathcal{N}_cM(G)||\gamma_{c+1}(G)| \le p^{\chi_{c+1}(n)}.$$

Theorem 1.3. [5] *Let G be an abelian group of order p^n. Then $|\mathcal{N}_cM(G)| = p^{\chi_{c+1}}(n)$ if and only if G is an elementary abelian p-group.*

Az a conculosion of Theorem 3.2 of [5] and 2.3 of [4] we have :

Theorem 1.4. [4,5] *Let $G \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus ... \oplus \mathbf{Z}_{n_d}$ be a finite d-generator p-group of order p^n and $\mathcal{V}$ be the variety of polynilpotent groups of a given class row, then*

$$|\mathcal{V}M(G)||V(G)| \le p^{f_n},$$

where $f_i = \chi_{c_t+1}(\chi_{c_{t-1}+1}(...(\chi_{c_1+1}(i))...))$ for all $1 \le i \le d$.

2. MAIN RESULTS

In a joint paper the second author [3], showed that if $G = \underbrace{\mathbf{Z} \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}}_{m-copies} \overset{n}{*} \mathbf{Z}_{r_1} \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}_{r_t}$ is the nth nilpotent product of some cyclic groups, where $c_1 \geq n$, r_{i+1} divides r_i for all $1 \leq i \leq t-1$ and $(p, r_1) = 1$ for all prime p less than or equal to n, then $\mathcal{N}_{c_1,c_2,\dots,c_s}M(G) = \mathbf{Z}^{(d_m)} \oplus \mathbf{Z}_{r_1}^{(d_{m+1}-d_m)} \oplus \dots \oplus \mathbf{Z}_{r_t}^{(d_{m+t}-d_{m+t-1})}$, where

$$d_i = \chi_{c_s+1}(\dots(\chi_{c_2+1}(\sum_{j=1}^{n} \chi_{c_1+j}(i)))\dots)$$

In this section, we use the structure of polynilpotent multipliers of nilpotent product to extend Theorem 1.3 to the polynilpotent multiplier of nilpotent products of cyclic p-groups with some conditions.

Theorem 2.1. *Let $G = \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$ be the nth nilpotent product of some cyclic groups, where $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_t$ and $(q, p) = 1$ for all prime q less than or equal to n. Let $\mathcal{N}_{c_1,c_2,\dots,c_s}$ be a variety of polynilpotent groups such that $c_1 \geq n$. Then $|\mathcal{N}_{c_1,c_2,\dots,c_s}M(G)| = p^{d_m}$ if and only if $G = \underbrace{\mathbf{Z}_p \overset{n}{*} \mathbf{Z}_p \overset{n}{*} \dots \overset{n}{*} \mathbf{Z}_p}_{m-copies}$, where $\sum_{i=1}^{t} \alpha_i = m$ and $d_m = \chi_{c_s+1}\dots(\chi_{c_2+1}(\sum_{j=1}^{n} \chi_{c_1+j}(m)\dots)$.*

With the assumption and notation of Theorem 2.1, let $n = 1$, then the nth nilpotent product of $\mathbf{Z}_{p^{\alpha_i}}$ $(1 \leq i \leq t)$ is the direct product of $\mathbf{Z}_{p^{\alpha_i}}$. So G is a finite abelian p-group of order p^m. Also d_i will be equal to f_i in Theorem 1.4. Therefore the following corollary is a consequence of the above Theorem.

Theorem 2.2. *Let G be an abelian group of order p^m. Then $|\mathcal{N}_{c_1,c_2,\dots,c_s}M(G)| = p^{f_m}$ if and only if G is an elementary abelian p-group*

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