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Wavelet transforms via generalized quasi-regular representations

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ABSTRACT

The construction of the well-known continuous wavelet transform has been extended before to higher dimensions. Then it was generalized to a group which is topologically isomorphic to a homogeneous space of the semidirect product of an abelian locally compact group and a locally compact group. In this paper, we consider a more general case. We introduce a class of continuous wavelet transforms obtained from the generalized quasi-regular representations. To define such a representation of a group G, we need a homogeneous space with a relatively invariant Radon measure and a character of G.

1. Introduction

A *wavelet* is known as a square integrable function, $\psi \in L^2(\mathbb{R})$, which satisfies the condition $C_\psi = \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\, d\omega < +\infty$, where $\hat{\psi}$ is obtained from ψ by the Plancherel theorem. A continuous wavelet transform (1-D CWT) of a function $f \in L^2(\mathbb{R})$ has been defined by using a wavelet ψ as follows

$$W_\psi f(b,a) = \frac{1}{\sqrt{|a|}} \int_{\mathbb{R}} f(t) \overline{\psi\left(\frac{t-b}{a}\right)}\, dt,$$

for almost all $a, b \in \mathbb{R}$, $a \neq 0$. The continuous wavelet transforms have been extended to higher dimensions (cf. [3, Chapter 9]). The n-dimensional continuous wavelet transforms can be obtained through the action of a Lie group G, which can be represented in the semidirect product form of two special Lie groups K and H, via a unitary representation on $L^2(K)$. In this case, K can be considered as a homogeneous space that G acts on it, and possesses a relatively invariant Radon measure.

In [2] it has been mentioned that how the continuous wavelet transforms can be defined on the two-sphere, a two-sheeted hyperboloid, and some other similar manifolds. Also, one may find in [5–7] some other aspects of generalization of continuous wavelet transforms related to homogeneous spaces.

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In [4] and [8] a more general case has been discussed; for a locally compact abelian group K and a locally compact group H, they have considered a continuous wavelet transform of the semidirect product $K \times' H$ on $L^2(K)$ defined by

$$\pi(k,h)f(x) = \sqrt{\frac{\Delta_G(h)}{\Delta_H(h)}}\, f\big(\tau_{h^{-1}}\big(k^{-1}x\big)\big) \quad (\text{almost all } x \in K),$$

where Δ_G and Δ_H are the modular functions on G and H, respectively. These kinds of wavelet transforms can be contained in a class of continuous wavelet transforms obtained from a continuous unitary representation of a locally compact group G on $L^2(S)$, where S is a homogeneous space that G acts on it. More specifically, when H is a closed subgroup of a locally compact group G, we define the generalized quasi-regular representations of G on $L^2(G/H)$. We show that these representations exist when G/H is attached to a relatively invariant Radon measure. In this case, all generalized quasi-regular representations can be exactly determined with the continuous homomorphisms defined from G into the unit circle $\mathbb{T}$. We consider the continuous wavelet transforms obtained from these representations as a class of wavelet transforms containing the above continuous wavelet transforms. To reach the goal we need first to prove some results on homogeneous spaces.

The outline of the rest of this paper is as follows: In Section 2, we introduce some notations and present some preliminary results on homogeneous spaces which we need in the sequel. In Section 3, we define a continuous unitary representation of G by using its action on a given homogeneous space, a generalized quasi-regular representation, which plays the role of the left regular representation of G. In Theorem 3.1 a necessary and sufficient condition is offered to have such a representation. We characterize this kind of representations of G in Theorem 3.3. In the last section, we consider the class of continuous wavelet transforms that can be obtained from a generalized quasi-regular representation in a similar way to what has been done in [10]. Finally, with a few examples, we show that in this manner we can get the n-dimensional continuous wavelet transforms and also the continuous wavelet transforms on a group which is topologically isomorphic to a homogeneous space of the semidirect product of an abelian locally compact group and a locally compact group, which have been discussed in [4,8]. Also, we offer some examples of continuous wavelet transforms that cannot be obtained by the way which has been introduced in [4,8].

2. Conditions for the existence of relatively invariant measures

Throughout this paper, when X is a locally compact Hausdorff space with a Radon measure μ, $C_c(X)$ consists of all continuous complex-valued functions on X with compact supports. Also, for each $1 \leqslant p < +\infty$, let $(L^p(X), \|\cdot\|_p)$ stand for the Banach space of equivalence classes of μ-measurable complex-valued functions on X whose pth powers are integrable. Moreover, G denotes a locally compact group with identity e, left Haar measure dx, and modular function Δ_G. Furthermore, we assume that H is a closed subgroup of G, then G/H is considered as a homogeneous space, and $q : G \to G/H$ denotes the canonical map. It is well known that $C_c(G/H)$ consists of all Pf functions, where f is a continuous function on G with compact support and

$$Pf(xH) = \int_H f(x\xi)\,d\xi \tag{1}$$

(cf. [9, Subsection 2.6]). Through the following lemma we illustrate a property of continuous functions with compact supports on homogeneous spaces, which will be required in Theorem 3.2. It is derived evidently in a similar way on groups.

Lemma 2.1. *For each $\varphi \in C_c(G/H)$ and $\varepsilon > 0$ there is a neighborhood V of e such that if $x, y \in G$ and $yx^{-1} \in V$, then*

$$\big|\varphi(xH) - \varphi(yH)\big| < \varepsilon.$$

Definition 2.2. Suppose that μ is a Radon measure on G/H. For each $x \in G$ we define the translation μ_x of μ by $\mu_x(E) = \mu(xE)$, where E is a Borel subset of G/H. μ is said to be *G-invariant* if $\mu_x = \mu$ for all $x \in G$, and is said to be *strongly quasi-invariant* provided that there is a continuous function $\lambda : G \times G/H \to (0, +\infty)$ which satisfies

$$d\mu_x(yH) = \lambda(x, yH)\,d\mu(yH) \quad (x, y \in G).$$

If the functions $\lambda(x,\cdot)$ reduce to constants, then μ is called *relatively invariant* under G.

We consider a rho-function for the pair (G, H) as a continuous function $\rho : G \to (0, +\infty)$ for which

$$\rho(x\xi) = \frac{\Delta_H(\xi)}{\Delta_G(\xi)}\rho(x) \quad (x \in G,\ \xi \in H).$$

It is well known that (G, H) admits a rho-function and for each rho-function ρ there is a strongly quasi-invariant measure μ on G/H such that

$$\int_{G/H} Pf(xH)\,d\mu(xH) = \int_G f(x)\rho(x)\,dx \quad \big(f \in C_c(G)\big). \tag{2}$$

μ also satisfies

$$\frac{d\mu_x}{d\mu}(yH) = \frac{\rho(xy)}{\rho(y)} \quad (x, y \in G). \tag{3}$$

Every strongly quasi-invariant measure on G/H arises from a rho-function in this manner, and all of these measures are strongly equivalent (cf. [9, Subsection 2.6]). Moreover, G/H has a G-invariant Radon measure if and only if the constant function $\rho(x) = 1$, $x \in G$, is a rho-function for the pair (G, H), or equivalently $\Delta_G|_H = \Delta_H$ (cf. [9, Theorem 2.49]). The next lemma describes the condition under which G/H has a relatively invariant measure.

Lemma 2.3. *The existence of a homomorphism rho-function $\rho : G \to (0, +\infty)$, for the pair (G, H), is a necessary and sufficient condition for the existence of a relatively invariant measure on G/H. More precisely, every relatively invariant measure on G/H is a positive constant multiple of another one, which arises from a homomorphism rho-function.*

Proof. Let μ be the strongly quasi-invariant measure which arises from a rho-function ρ. If ρ is a homomorphism, then by (3) we get $d\mu_x = \rho(x)\, d\mu$, for all $x \in G$, which shows that μ is relatively invariant under G. Conversely, if μ is a relatively invariant measure, then there is a continuous function $\lambda : G \to (0, +\infty)$ such that $d\mu_x = \lambda(x)\, d\mu$, for all $x \in G$. So for each $x \in G$ and $f \in C_c(G)$ we can write

$$\begin{aligned}
\int_G f(y)\rho(xy)\, dy &= \int_G f\big(x^{-1}y\big)\rho(y)\, dy \\
&= \int_{G/H} P(L_x f)(yH)\, d\mu(yH) \\
&= \lambda(x) \int_{G/H} Pf(yH)\, d\mu(yH) \\
&= \lambda(x) \int_G f(y)\rho(y)\, dy.
\end{aligned}$$

Thus for a fixed $x \in G$ we have

$$\int_G f(y)\big(\rho(xy) - \lambda(x)\rho(y)\big)\, dy = 0,$$

for all $f \in C_c(G)$. This leads to

$$\frac{\rho(xy)}{\rho(y)} = \lambda(x) \quad (x, y \in G).$$

Now define $\rho_0 : G \to \mathbb{R}$ by $\rho_0(x) = \rho(x)/\rho(e)$. An easy calculation shows that ρ_0 is a homomorphism rho-function for (G, H). Also, $d\mu = \rho(e)\, d\mu_0$ where μ_0 is the relatively invariant Radon measure on G/H which arises from ρ_0. $\square$

In the next proposition, we show that if G is the semidirect product of two locally compact groups K and H respectively, then G/H has a relatively invariant Radon measure. For this, we need to fix some notations. Suppose that K and H are two locally compact groups with neutral elements e_K and e_H respectively. If $h \mapsto \tau_h$ is a homomorphism of H into the group of automorphisms of K and the mapping $(k, h) \mapsto \tau_h(k)$ from $K \times H$ onto K is continuous with respect to the usual product topology, then the set $K \times H$ endowed with the operations

$$(k_1, h_1) \cdot (k_2, h_2) = \big(k_1 \tau_{h_1}(k_2), h_1 h_2\big) \quad \text{and} \quad (k, h)^{-1} = \big(\tau_{h^{-1}}\big(k^{-1}\big), h^{-1}\big)$$

is a locally compact group, with neutral element (e_K, e_H), which is called the semidirect product of K and H respectively, and is denoted by $K \times' H$. From now on, the elements (k, e_H) and (e_K, h) will be represented by k and h, where $k \in K$ and $h \in H$. Because of the identity $(k, h) = (k, e_H) \cdot (e_K, h)$, by the abbreviation above, kh will be another presentation for the element (k, h) of $K \times' H$.

Proposition 2.4. *Let G be the semidirect product of two locally compact groups K and H respectively; i.e. $G = K \times' H$. Then G/H has a relatively invariant Radon measure, which may arise from the rho-function $\rho : G \to \mathbb{R}$ defined by $\rho(x) = \Delta_H(h)/\Delta_G(h)$, where $x \in G$, $h \in H$, and $x = kh$ for some $k \in K$.*

Proof. Define $\rho : G \to \mathbb{R}$ by $\rho(x) = \Delta_H(h)/\Delta_G(h)$, where $x = kh \in G$, $k \in K$, $h \in H$. It is easy to show that ρ is a rho-function for (G, H). Also, if $x = k_1 h_1$ and $y = k_2 h_2$, in which $k_1, k_2 \in K$ and $h_1, h_2 \in H$, then we have

$$\begin{aligned}\rho(xy) &= \rho(k_1h_1k_2h_2)\\ &= \rho(k_1k'h_1h_2)\\ &= \frac{\Delta_H(h_1h_2)}{\Delta_G(h_1h_2)}\\ &= \rho(x)\cdot\rho(y),\end{aligned}$$

for some $k' \in K$. Then it follows that ρ is a homomorphism rho-function for (G, H). □

3. Criteria for the construction of generalized quasi-regular representations

From now on, by "a representation" $(\pi, \mathcal{H})$ of a locally compact group G we mean "a continuous unitary representation"; i.e. a homomorphism π from G to the group of unitary operators on Hilbert space $\mathcal{H}$, $U(\mathcal{H})$, where $U(\mathcal{H})$ is equipped with the strong operator topology. Also, for a function f on G and for each $x \in G$, the left translation of f by x is defined by $L_x f(y) = f(x^{-1}y)$, $y \in G$. By using the action of a group G on itself, one can define a representation $\pi : G \to U(L^2(G))$, *the left regular representation*, as follows:

$$\pi(x)(f) = L_x f \quad \big(x \in G,\ f \in L^2(G)\big).$$

For a closed subgroup H of G, if G/H has a G-invariant Radon measure μ which arises from the constant function $\rho(x) = 1$, $x \in G$, then there exists a representation $\pi : G \to U(L^2(G/H))$, *the quasi-regular representation*, such that

$$\pi(x)(Pf) = P(L_x f) \quad \big(x \in G,\ f \in C_c(G)\big),$$

where Pf satisfies (1). In other words,

$$\pi(x)\varphi(yH) = \varphi\big(x^{-1}yH\big) \quad \left(\mu\text{-almost all } yH \in \frac{G}{H}\right),$$

where $x \in G$, $\varphi \in L^2(G/H)$ (cf. [9, Subsection 6.1]). With this representation, G acts on $L^2(G/H)$ in the same way that G acts on $L^2(G)$ by the left regular representation. Obviously, in the special case that H is the trivial subgroup $\{e\}$, the quasi-regular representation coincides with the left regular representation.

In the general case, via each $x \in G$, define

$$\begin{cases} \pi(x) : L^2\left(\frac{G}{H}\right) \to L^2\left(\frac{G}{H}\right),\\ \pi(x)\varphi(yH) = h(x)\varphi\big(x^{-1}yH\big) \end{cases}$$

for μ-almost all $yH \in G/H$, in which $h(x)$ is a complex number. Also for each $\varphi \in C_c(G/H)$, take $f \in C_c^+(G)$ so that $|\varphi|^2 = Pf$. Then

$$\begin{aligned}\big\|\pi(x)\varphi\big\|_2^2 &= \int_{G/H} \big|\pi(x)\varphi(yH)\big|^2\, d\mu(yH)\\ &= \int_{G/H} \big|h(x)\big|^2 Pf\big(x^{-1}yH\big)\, d\mu(yH)\\ &= \int_G f\big(x^{-1}y\big)\big|h(x)\big|^2\rho(y)\, dy\\ &= \int_G f(y)\big|h(x)\big|^2\rho(xy)\, dy.\end{aligned}$$

Since $\|\varphi\|_2^2 = \int_G f(y)\rho(y)\,dy$, $\pi(x)$ will be unitary if and only if

$$\int_G f(y)\big(\big|h(x)\big|^2\rho(xy) - \rho(y)\big)\, dy = 0,$$

for all $f \in C_c^+(G)$. This leads to $|h(x)|^2\rho(xy) = \rho(y)$ for all $y \in G$. So $\pi(x)$ will be unitary if and only if $\frac{\rho(y)}{\rho(xy)} = |h(x)|^2$ only depends on x. It follows that we can define such a unitary representation π if and only if μ is relatively invariant. The argument above can be summarized as follows:

Theorem 3.1. *The existence of a homomorphism rho-function for the pair (G, H) is a necessary and sufficient condition to have a representation $\pi : G \to U(L^2(G/H))$, with*

$$\pi(x)\varphi(yH) = h(x)\varphi\big(x^{-1}yH\big) \quad \left(\mu\text{-almost all } yH \in \frac{G}{H}\right) \tag{4}$$

for some constants $h(x) \in \mathbb{C}$. In other words, there exists such a representation $\pi : G \to U(L^2(G/H))$ if and only if the measure μ on G/H is relatively invariant.

We call a representation $\pi : G \to U(L^2(G/H))$ which satisfies (4) a generalized quasi-regular representation of G. The next theorem connects the generalized quasi-regular representations to the continuous homomorphisms defined from G into the multiplicative group $(\mathbb{C} - \{0\}, \cdot)$.

Theorem 3.2. *Consider $(G/H, \mu)$ as a measure space with relatively invariant measure μ that arises from a rho-function ρ. Then a representation $\pi : G \to U(L^2(G/H))$ which satisfies (4) can be specified precisely with a continuous homomorphism $h : G \to (\mathbb{C} - \{0\}, \cdot)$ for which $|h(x)| = \sqrt{\rho(e)/\rho(x)}$, $x \in G$.*

Proof. First suppose that π is a representation of G on $L^2(G/H)$ which satisfies the equality (4). Then by Theorem 3.1, $|h(x)| = \sqrt{\rho(e)/\rho(x)}$ for all $x \in G$. Also, for each $x, y, z \in G$ and $\varphi \in C_c(G/H)$ we have

$$\begin{aligned} h(xy)\varphi\big(y^{-1}x^{-1}zH\big) &= \pi(xy)\varphi(zH) \\ &= \big(\pi(x)\pi(y)\big)\varphi(zH) \\ &= h(x)\big(\pi(y)\varphi\big)\big(x^{-1}zH\big) \\ &= h(x)h(y)\varphi\big(y^{-1}x^{-1}zH\big). \end{aligned}$$

It turns out that the scalars $h(x)$, $x \in G$, define a homomorphism $h : G \to (\mathbb{C} - \{0\}, \cdot)$. To show that h is continuous, assume that a net $\{x_\alpha\}_{\alpha \in I}$ tends to e in G. Take an open neighborhood U of e with compact closure. Suppose that $x_\alpha \in U$ for all $\alpha \in I$, and let φ be a nonzero continuous function on G/H whose support is contained in $q(U)$. Then $\pi(x_\alpha)\varphi \to \varphi$ in $L^2(G/H)$; i.e.

$$\int_{G/H} \big|\pi(x_\alpha)\varphi(yH) - \varphi(yH)\big|^2 \, d\mu(yH) \to 0,$$

as $x_\alpha \to e$. Since $\operatorname{supp}(\pi(x_\alpha)\varphi - \varphi) \subseteq q(U^2)$, $\alpha \in I$, it follows from Hölder's inequality that

$$\int_{G/H} \big|h(x_\alpha)\varphi\big(x_\alpha^{-1}yH\big) - \varphi(yH)\big| \, d\mu(yH) \to 0 \tag{5}$$

as $x_\alpha \to e$. Because of the identity $|h(x_\alpha)| = \sqrt{\rho(e)/\rho(x_\alpha)}$ we get $|h(x_\alpha)| \to 1$ as $x_\alpha \to e$. In addition, by Lemma 2.1, we get

$$\int_{G/H} \big|\varphi\big(x_\alpha^{-1}yH\big) - \varphi(yH)\big| \, d\mu(yH) \to 0 \quad \text{as } x_\alpha \to e. \tag{6}$$

Now, we can write

$$\begin{aligned} \big|h(x_\alpha) - 1\big| \|\varphi\|_1 &= \int_{G/H} \big|h(x_\alpha)\varphi(yH) - \varphi(yH)\big| \, d\mu(yH) \\ &\leqslant \big|h(x_\alpha)\big| \int_{G/H} \big|\varphi\big(x_\alpha^{-1}yH\big) - \varphi(yH)\big| \, d\mu(yH) + \int_{G/H} \big|h(x_\alpha)\varphi\big(x_\alpha^{-1}yH\big) - \varphi(yH)\big| \, d\mu(yH) \end{aligned}$$

which shows that $h(x_\alpha) \to 1$, by (5) and (6), where $\{x_\alpha\}$ approaches e.

For the converse, suppose that $h : G \to (\mathbb{C} - \{0\}, \cdot)$ is a continuous homomorphism which satisfies $|h(x)| = \sqrt{\rho(e)/\rho(x)}$ where $x \in G$. Define $\pi(x) : L^2(G/H) \to L^2(G/H)$, via each $x \in G$, by

$$\pi(x)\varphi(yH) = h(x)\varphi\big(x^{-1}yH\big) \quad \left(\mu\text{-almost all } yH \in \frac{G}{H}\right). \tag{7}$$

Then for each $x, y \in G$, $|h(x)|^2 = \frac{\rho(y)}{\rho(xy)}$ only depends on x and so $\pi(x)$, $x \in G$, will be unitary on $L^2(G/H)$. The equality $\pi(xy) = \pi(x)\pi(y)$, $x, y \in G$, results from the assumption that h is a homomorphism. Moreover, for each $\varphi \in L^2(G/H)$, $\{\pi(x_\alpha)\varphi\}_{\alpha \in I}$ approaches φ in $L^2(G/H)$, where $\{x_\alpha\}_{\alpha \in I}$ tends to e in G. It suffices to prove this for each $\varphi \in C_c(G/H)$.

Let $\varphi \in C_c(G/H)$ be a nonzero function and let $\{x_\alpha\}_{\alpha\in I}$ approach e in G. Then there is a continuous function f with compact support K for which $\varphi = Pf$ and $\operatorname{supp}(\varphi) = q(K)$. Fix an open neighborhood V of e with compact closure and put $M = \sup\{|h(x)|:\ x \in V\}$. By Lemma 2.1, for a given $\varepsilon > 0$ there exists a neighborhood U of e such that $U \subseteq V$ and

$$\left|\varphi\left(x^{-1}yH\right) - \varphi(yH)\right| < \frac{\varepsilon}{s} \quad (x \in U,\ \ y \in G),$$

where $s = \sqrt{\|\varphi\|_2^2 + 2M\|\varphi\|_1 + M^2 q(VK)}$. Take an $\alpha_0 \in I$ for which $x_\alpha \in U$ and $|h(x_\alpha) - 1| < \varepsilon/s$ provided that $\alpha \geqslant \alpha_0$. Then $\operatorname{supp}(\pi(x_\alpha)\varphi - \varphi) \subseteq q(VK)$, for all $\alpha \in I$, and hence we get

$$\begin{aligned}
\left\|\pi(x_\alpha)\varphi - \varphi\right\|_2^2 &= \int_{G/H} \left|h(x_\alpha)\varphi\left(x_\alpha^{-1}yH\right) - \varphi(yH)\right|^2 d\mu(yH) \\
&\leqslant \int_{G/H} \left(\left|h(x_\alpha)\right|\left|\varphi\left(x_\alpha^{-1}yH\right) - \varphi(yH)\right| + \left|h(x_\alpha) - 1\right|\left|\varphi(yH)\right|\right)^2 d\mu(yH) \\
&= \int_{q(VK)} \left(\left|h(x_\alpha)\right|\left|\varphi\left(x_\alpha^{-1}yH\right) - \varphi(yH)\right| + \left|h(x_\alpha) - 1\right|\left|\varphi(yH)\right|\right)^2 d\mu(yH) \\
&< \frac{\varepsilon^2}{s^2} \int_{q(VK)} \left(M + \left|\varphi(yH)\right|\right)^2 d\mu(yH) \\
&= \varepsilon^2,
\end{aligned}$$

for all $\alpha \geqslant \alpha_0$. Therefore, $\pi : G \to U(L^2(G/H))$ defined by (7) is a representation of G on $L^2(G/H)$. $\square$

Let H be a closed subgroup of G and η be a unitary representation of H. We mean by $\operatorname{Ind}_H^G(\eta)$ the unitary representation of G which is induced by η (cf. [9, Subsection 6.1]). We also recall that a character of G is a continuous homomorphism from G to the unit circle $\mathbb{T}$. Through the next theorem we show that each of the generalized quasi-regular representations of G is the tensor product of a character of G and the induced representation $\operatorname{Ind}_H^G 1$, where 1 is the trivial representation of H on $\mathbb{C}$.

Theorem 3.3. *Let μ be a relatively invariant Radon measure on G/H which arises from a rho-function ρ. Then a generalized quasi-regular representation of G can be uniquely determined by a character of G. More precisely, all generalized quasi-regular representations of G can be written in the form $\sigma \otimes \pi_0$, in which $\sigma : G \to \mathbb{T}$ is a character of G and π_0 is the induced representation* $\operatorname{Ind}_G^H 1$.

Proof. Let $\pi : G \to U(L^2(G/H))$ be a generalized quasi-regular representation of G. According to Theorem 3.2, there exists a continuous homomorphism $h : G \to (\mathbb{C} - \{0\}, \cdot)$ so that for each $x \in G$, $|h(x)| = \sqrt{\rho(e)/\rho(x)}$ and

$$\pi(x)\varphi(yH) = h(x)\varphi\left(x^{-1}yH\right) \quad \left(\mu\text{-almost all } yH \in \frac{G}{H}\right),$$

where $\varphi \in L^2(G/H)$. We can write $h = \sigma|h|$ in which $\sigma : G \to \mathbb{T}$ is a character of G. If π_0 is the induced representation $\operatorname{Ind}_H^G 1$ of G, then

$$\pi_0(x)\varphi(yH) = \left|h(x)\right|\varphi\left(x^{-1}yH\right) \quad \left(\mu\text{-almost all } yH \in \frac{G}{H}\right),$$

where $x \in G$ and $\varphi \in L^2(G/H)$ (cf. [9, Subsection 6.1]). Let $\pi_\sigma : G \to U(\mathbb{C})$ be the representation defined by σ; i.e. $\pi_\sigma(x) = \sigma(x)$, $x \in \mathbb{C}$. For simplicity, we denote this representation by σ. For each $x \in G$ the following diagram commutes:

$$\begin{tikzcd}
L^2\left(\frac{G}{H}\right) \arrow[r, "\pi(x)"] \arrow[d, "\mathrm{ins}"'] & L^2\left(\frac{G}{H}\right) \arrow[d, "\mathrm{ins}"] \\
\mathbb{C} \otimes L^2\left(\frac{G}{H}\right) \arrow[r, "(\sigma\otimes\pi_0)(x)"'] & \mathbb{C} \otimes L^2\left(\frac{G}{H}\right)
\end{tikzcd}$$

where $\mathrm{ins} : L^2(G/H) \to \mathbb{C} \otimes L^2(G/H)$ is the unitary defined by $\mathrm{ins}(\varphi) = 1 \otimes \varphi$. Since for each $\varphi \in L^2(G/H)$ we have

$$\begin{aligned}
\left(\mathrm{ins}\, o\pi(x)\right)\varphi(\lambda \otimes yH) &= \left(1 \otimes \pi(x)\varphi\right)(\lambda \otimes yH) \\
&= \lambda \otimes \left(\pi(x)\varphi(yH)\right)
\end{aligned}$$

$$\begin{aligned}
&= \lambda \otimes \sigma(x)\big|h(x)\big|\varphi\big(x^{-1}yH\big)\\
&= \sigma(x)\lambda \otimes \pi_0(x)\varphi(yH)\\
&= \big(\sigma(x) \otimes \pi_0(x)\big)(1 \otimes \varphi)(\lambda \otimes yH)\\
&= \big((\sigma \otimes \pi_0)(x) o \operatorname{ins}\big)\varphi(\lambda \otimes yH),
\end{aligned}$$

for all $\lambda \in \mathbb{C}$ and for μ-almost all $yH \in G/H$. Therefore, $\operatorname{ins} o \pi(x) = (\sigma \otimes \pi_0)(x) o \operatorname{ins}$, for all $x \in G$, and hence $\pi \cong \sigma \otimes \pi_0$.

For the reverse direction, let σ be a character of G. Then $h : G \to (\mathbb{C} - \{0\}, \cdot)$ is a continuous homomorphism, where $h(x) = \sigma(x)\sqrt{\rho(e)/\rho(x)}$, $x \in G$. Theorem 3.2 guarantees that $\pi : G \to U(L^2(G/H))$ is a representation of G where

$$\pi(x)\varphi(yH) = \sigma(x)\sqrt{\frac{\rho(e)}{\rho(x)}}\varphi\big(x^{-1}yH\big) \quad \left(\mu\text{-almost all } yH \in \frac{G}{H}\right).$$

Trivially, $|h(x)| = \sqrt{\rho(e)/\rho(x)}$ for all $x \in G$, $h = \sigma|h|$, and by the first part of the proof we have $\pi \cong \sigma \otimes \pi_0$. $\square$

4. Conclusion and examples

Suppose that π is a representation of G on a Hilbert space $\mathcal{H}$. A *subrepresentation* of π is a pair (M, π^M) in which M is a π-invariant closed subspace of $\mathcal{H}$ and π^M is the restriction of π to M. An irreducible representation π of G on $\mathcal{H}$ is *square integrable* if there exist two nonzero vectors $u, v \in \mathcal{H}$ for which the function on G defined almost everywhere by $x \mapsto \langle u, \pi(x)v\rangle$ is square integrable. Then v is called *an admissible vector*. When $(\pi, \mathcal{H})$ is a square integrable representation of G, we can define the continuous wavelet transforms in the same way to what has been done in [10]. More specifically, by using an admissible vector v, a continuous wavelet transform W_v is introduced as the isometry $W_v : \mathcal{H} \to L^2(G)$, defined by

$$W_v u(x) = \frac{1}{\sqrt{C_v}}\big\langle u, \pi(x)v\big\rangle \quad (\text{almost all } x \in G),$$

in which C_v is a positive real number satisfying

$$C_v = \frac{1}{\|v\|^2}\int_G \big|\big\langle v, \pi(x)v\big\rangle\big|^2\,dx.$$

In addition, we can reconstruct an element u of $\mathcal{H}$ via its image $W_v u$ as follows:

$$u = \frac{1}{\sqrt{C_v}}\int_G (W_v u)(x)\pi(x)v\,dx, \tag{8}$$

where the equality holds in the weak-sense. In other words,

$$I_{\mathcal{H}} = \frac{1}{C_v}\int_G \big\langle \cdot, \pi(x)v\big\rangle\pi(x)v\,dx.$$

Let $(G/H, \mu)$ be a measure space with a relatively invariant Radon measure μ and let $\pi : G \to U(L^2(G/H))$ be a generalized quasi-regular representation which has a square integrable subrepresentation (π^M, M). We consider the continuous wavelet transforms arising from (π^M, M) as above; i.e.

$$\begin{cases} W_\psi : M \to L^2(G),\\ W_\psi\varphi(x) = \dfrac{1}{\sqrt{C_\psi}}\big\langle\varphi, \pi(x)\psi\big\rangle. \end{cases}$$

If ρ is a homomorphism rho-function for the pair (G, H), we can define a generalized quasi-regular representation $\pi : G \to U(L^2(G/H))$ by

$$\pi(x)\varphi(yH) = \frac{1}{\sqrt{\rho(x)}}\varphi\big(x^{-1}yH\big).$$

Then $\pi = \operatorname{Ind}_H^G 1$ and we have a class of continuous wavelet transforms provided that π has a square integrable subrepresentation. The following examples show that the n-dimensional continuous wavelet transforms can be obtained in this way. Also, when $G = K \times' H$ for some locally compact abelian group K, we get the continuous wavelet transforms on $L^2(K)$ which have been discussed in [4,8]. Furthermore, we offer some examples of continuous wavelet transforms of a group G on $L^2(K)$, by using a generalized quasi-regular representation, where K is a closed subgroup of G such that $L^2(K) \cong L^2(G/H)$ for some unimodular closed subgroup H of G, but $G \ncong K \times' H$.

Note that if H is a unimodular closed subgroup of a group G, then the function $\rho : G \to \mathbb{R}$ defined by $\rho(x) = \Delta_G(x)^{-1}$ is a homomorphism rho-function for the pair (G, H). So there is a generalized quasi-regular representation $\pi : G \to U(L^2(G/H))$ which satisfies

$$\pi(x)\varphi(yH) = \sqrt{\Delta_G(x)}\varphi\bigl(x^{-1}yH\bigr) \quad \left(\mu\text{-almost all } yH \in \frac{G}{H}\right).$$

One can easily check that every continuous function of G/H with compact support is an admissible vector for π provided that H is a compact subgroup of G. So we can define a continuous wavelet transform by using a square integrable subrepresentation of π.

Example 4.1 *(n-D CWT).* Fix an element $\psi \in L^2(\mathbb{R}^n)$ with condition

$$\int_{\mathbb{R}^n} \frac{|\hat{\psi}(\boldsymbol{k})|}{|\boldsymbol{k}|^n} d^n\boldsymbol{k} < +\infty. \tag{9}$$

The n-dimensional continuous wavelet transform of an element $f \in L^2(\mathbb{R}^n)$ has been defined, as a function on $\mathbb{R}^n \times \mathbb{R}^+ \times \mathrm{SO}(n)$, by

$$W_\psi f(\boldsymbol{b}, a, \omega) = a^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\boldsymbol{x})\overline{\psi\bigl(a^{-1}\omega^{-1}(\boldsymbol{x}-\boldsymbol{b})\bigr)}\, d^n\boldsymbol{x},$$

where $\mathrm{SO}(n)$ indicates the group of rotations around the origin of $\mathbb{R}^n$ (cf. [3, Chapter 9]). Evidently, $\mathbb{R}^n$ can be considered as a homogeneous space of the similitude group, $\mathrm{SIM}(n) = \mathbb{R}^n \times' (\mathbb{R}^+ \times \mathrm{SO}(n))$. Moreover, the function $\rho : \mathrm{SIM}(n) \to (0, +\infty)$, defined by $\rho(\boldsymbol{b}, a, \omega) = a^n$, is a homomorphism rho-function for $(\mathrm{SIM}(n), \mathbb{R}^+ \times \mathrm{SO}(n))$. Therefore, we can define a unitary representation $\pi : \mathrm{SIM}(n) \to U(L^2(\mathbb{R}^n))$ by

$$\pi(\boldsymbol{b_0}, a_0, \omega_0)\psi(\boldsymbol{b}) = a_0^{-n/2}\psi\bigl(a_0^{-1}\omega_0^{-1}(\boldsymbol{b} - \boldsymbol{b_0})\bigr).$$

This representation is square integrable and (9) is the admissibility condition of a vector ψ (cf. [3]). An easy calculation shows that

$$W_\psi f = \bigl\langle f, \pi(\boldsymbol{b}, a, \omega)\psi \bigr\rangle.$$

Therefore, by (8), we can reconstruct an element $f \in L^2(\mathbb{R}^n)$ from $W_\psi f$ as follows:

$$\begin{aligned} f(\boldsymbol{x}) &= \frac{1}{C_\psi} \int_{\mathbb{R}^n}\int_{\mathbb{R}^+}\int_{\mathrm{SO}(n)} (W_\psi f)(\boldsymbol{b}, a, \omega)\pi(\boldsymbol{b}, a, \omega)\psi(\boldsymbol{x}) \frac{1}{a|\omega|^n}\, d\omega\, da\, d\boldsymbol{b} \\ &= \frac{1}{C_\psi} \int_{\mathbb{R}^n}\int_{\mathbb{R}^+}\int_{\mathrm{SO}(n)} (W_\psi f)(\boldsymbol{b}, a, \omega)\psi\bigl(a^{-1}\omega^{-1}(\boldsymbol{x}-\boldsymbol{b})\bigr) \frac{1}{a^{1+\frac{n}{2}}|\omega|^n}\, d\omega\, da\, d\boldsymbol{b}, \end{aligned}$$

where ψ is an admissible vector and the equalities hold for almost all $\boldsymbol{x} \in \mathbb{R}^n$.

Example 4.2. Let K and H be two locally compact groups and $G = K \times' H$. By Proposition 2.4, the pair (G, H) admits a homomorphism rho-function ρ such that $\rho(x) = \Delta_H(h)/\Delta_G(h)$, where $x = kh$, $k \in K$, and $h \in H$. Therefore, we can offer a continuous wavelet transform via the representation $\pi : G \to U(L^2(G/H))$, defined by

$$\pi(x)\varphi(yH) = \sqrt{\frac{\Delta_G(h)}{\Delta_H(h)}}\varphi\bigl(x^{-1}yH\bigr) \quad \left(\mu\text{-almost all } yH \in \frac{G}{H}\right),$$

where $k \in K$, $h \in H$, and $x = kh$. Since $x^{-1}y = (\tau_{h^{-1}}(k^{-1}k_1), h^{-1}h_1)$ for all $y = k_1h_1$, $k_1 \in K$, and $h_1 \in H$ and $L^2(G/H)$ is isometrically isomorphic to $L^2(K)$, we can redefine the representation π as follows:

$$\begin{cases} \pi : G \to U\bigl(L^2(K)\bigr), \\ \pi(x)\varphi(k_1) = \sqrt{\dfrac{\Delta_G(h)}{\Delta_H(h)}}\varphi\bigl(\tau_{h^{-1}}\bigl(k^{-1}k_1\bigr)\bigr) \end{cases}$$

for almost all $k_1 \in K$ where $k \in K$, $h \in H$, and $x = kh$. The continuous wavelet transform which is obtained via this representation is the same wavelet transform that has been defined in [8], where K is abelian, and its admissibility condition has been discussed in [4].

Example 4.3. Let N be an even number and G be the group with two generators a and b satisfying $|a| = N$, $|b| = 3$, and $ba = ab^2$; i.e. $G = \langle a, b\colon\ a^N = b^3 = e,\ ba = ab^2\rangle$. Also, let $H = \langle b\rangle$, $K = \langle a\rangle$, and $\pi : G \to U(L^2(G/H))$ be the quasi-regular representation; i.e.

$$\begin{aligned}\pi\big(a^i b^j\big)\varphi\big(a^n H\big) &= \varphi\big(b^{-j} a^{n-i} H\big)\\ &= \varphi\big(a^{n-i} H\big),\end{aligned}$$

for all $0 \leqslant i, n \leqslant N - 1$ and $0 \leqslant j \leqslant 2$. It is easy to check that $L^2(G/H)$ is isometrically isomorphic to $L^2(K)$ and hence we can rewrite the quasi-regular representation π as follows:

$$\begin{cases}\pi : G \to U\big(L^2(K)\big),\\ \pi\big(a^i b^j\big) f\big(a^n\big) = f\big(a^{n-i}\big).\end{cases}$$

It easy to see that every irreducible subrepresentation (π^M, M) of π is square integrable and every function $g \in M$ is an admissible vector. In this case M should be one dimensional. For instance, if M is the closed subspace of $L^2(K)$ generated by $f_0 \in L^2(K)$, where $f_0(a^n) = (-1)^n$ for all $0 \leqslant n \leqslant N - 1$, then (π^M, M) is a square integrable subrepresentation of π. Therefore, we can define a continuous wavelet transform $W_g : M \to L^2(G)$, associated to a function $g \in M$, by

$$W_g f\big(a^i b^j\big) = \frac{1}{\sqrt{C_g}} \sum_{n=0}^{N-1} f\big(a^n\big)\overline{g\big(a^{n-i}\big)} \quad (0 \leqslant i \leqslant N - 1,\ 0 \leqslant j \leqslant 2),$$

where

$$C_g = \frac{3}{\|g\|_2^2} \sum_{i=0}^{N-1} \left| \sum_{n=0}^{N-1} g\big(a^n\big)\overline{g\big(a^{n-i}\big)} \right|^2.$$

Moreover, for all function $f \in M$ we have

$$f\big(a^n\big) = \frac{1}{\sqrt{C_g}} \sum_{i=0}^{N-1} \sum_{j=0}^{2} W_g f\big(a^i b^j\big) g\big(a^{n-i}\big),$$

for all $0 \leqslant n \leqslant N - 1$. Note that, in this example, $G \ncong K \times' H$ since K is not a normal subgroup of G. But H is a closed normal subgroup of G and this causes that $W_g f$ is constant on the left cosets of H; i.e. $W_g f(a^i b^j)$ does not depend on j, for all $0 \leqslant i \leqslant N - 1$.

In the next example, we offer a group G with two closed subgroups H and K that none of them is normal in G, $G = HK$, and $H \cap K = \langle e\rangle$. Then by using an irreducible subrepresentation of the quasi-regular representation of G on $L^2(G/H)$ we can define a continuous wavelet transform. In this case, the continuous wavelet transform of a function is not constant on the left cosets of H.

Example 4.4. Suppose that N is an odd number with $N \geqslant 5$. Take $G = A_N$, the alternating group of degree N, $H = A_{N-1}$, and $K = \langle\alpha\rangle$ where $\alpha \in A_N$ is the N-cycle $(1\ 2 \ldots N) \in A_N$. Then G is a simple group and none of the closed subgroups H and K is normal in G. Furthermore, $G = HK$ and $H \cap K = \langle e\rangle$ (cf. [1]). More precisely, for all $\gamma \in G$ there exists a unique $\beta \in H$ such that $\gamma = \alpha^i\beta$ in which $i = \gamma(N)$. Therefore,

$$\gamma H = \alpha^{\gamma(N)} H \quad (\gamma \in G). \tag{10}$$

Now, let $\pi : G \to U(L^2(G/H))$ be the quasi-regular representation. Also, let $\psi \in L^2(G/H)$ and $\zeta \in G$. Then there exists some $\beta \in H$ and $1 \leqslant i \leqslant N$ for which $\zeta = \alpha^i\beta$. By using (10), for all $1 \leqslant n \leqslant N$ we get

$$\begin{aligned}\pi(\zeta)\psi\big(\alpha^n H\big) &= \psi\big(\beta^{-1}\alpha^{n-i} H\big)\\ &= \psi\big(\alpha^{\beta^{-1}\alpha^{n-i}(N)} H\big)\\ &= \begin{cases}\psi(\alpha^{\beta^{-1}(N+n-i)} H), & n \leqslant i,\\ \psi(\alpha^{\beta^{-1}(n-i)} H), & i < n.\end{cases}\end{aligned}$$

As it has been done in Example 4.3, we can rewrite the representation π of G as $\pi : G \to U(L^2(K))$, where

$$\pi\big(\alpha^i\beta\big) g\big(\alpha^n\big) = \begin{cases} g(\alpha^{\beta^{-1}(N+n-i)}), & n \leqslant i,\\ g(\alpha^{\beta^{-1}(n-i)}), & i < n,\end{cases}$$

for all $1 \leqslant i, n \leqslant N$ and $\beta \in H$. Moreover, by using an irreducible subrepresentation (π^M, M) of $\pi : G \to U(L^2(K))$ and a function $g \in M$, we can define a continuous wavelet transform $W_g : M \to U(L^2(K))$, by

$$W_g f(\alpha^i \beta) = \frac{1}{\sqrt{C_g}} \left(\sum_{n=1}^{i} f(\alpha^n) \overline{g(\alpha^{\beta^{-1}(N+n-i)})} + \sum_{n=i+1}^{N} f(\alpha^n) \overline{g(\alpha^{\beta^{-1}(n-i)})} \right),$$

where $1 \leqslant i \leqslant N$, $\beta \in H$, and

$$C_g = \frac{1}{\|g\|_2^2} \sum_{\beta \in A_{N-1}} \sum_{i=1}^{N} \left| \sum_{n=1}^{i} g(\alpha^n) \overline{g(\alpha^{\beta^{-1}(N+n-i)})} + \sum_{n=i+1}^{N} g(\alpha^n) \overline{g(\alpha^{\beta^{-1}(n-i)})} \right|^2.$$

Also, for all $f \in M$ we have

$$f(a^n) = \frac{1}{\sqrt{C_g}} \sum_{\beta \in A_{N-1}} \left(\sum_{i=1}^{n-1} W_g f(\alpha^i \beta) g(\alpha^{\beta^{-1}(n-i)}) + \sum_{i=n}^{N} W_g f(\alpha^i \beta) g(\alpha^{\beta^{-1}(N+n-i)}) \right),$$

where $1 \leqslant n \leqslant N$.

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