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On the Order of Polynilpotent Multipliers of Some Nilpotent Products of Cyclic p-Groups

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Abstract. In this article we show that if $\mathcal{V}$ is the variety of polynilpotent groups of class row $(c_1, c_2, ..., c_s)$, $\mathcal{N}_{c_1,c_2,...,c_s}$, and $G \cong \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$ is the nth nilpotent product of some cyclic p-groups, where $c_1 \geq n$, $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_t$ and $(q,p) = 1$ for all primes q less than or equal to n, then $|\mathcal{N}_{c_1,c_2,...,c_s} M(G)| = p^{d_m}$ if and only if $G \cong \mathbf{Z}_p \overset{n}{*} \mathbf{Z}_p \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_p$ (m-copies), where $m = \sum_{i=1}^{t} \alpha_i$ and $d_m = \chi_{c_s+1}(...(\chi_{c_2+1}(\sum_{j=1}^{n} \chi_{c_1+j}(m)))...)$. Also, we extend the result to the multiple nilpotent product $G \cong \mathbf{Z}_{p^{\alpha_1}} \overset{n_1}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n_2}{*} ... \overset{n_{t-1}}{*} \mathbf{Z}_{p^{\alpha_t}}$, where $c_1 \geq n_1 \geq ... \geq n_{t-1}$. Finally a similar result is given for the c-nilpotent multiplier of $G \cong \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$ with the different conditions $n \geq c$ and $(q,p) = 1$ for all primes q less than or equal to $n + c$.

Keywords: Polynilpotent multiplier; Nilpotent product; Cyclic group; Finite p-group; Elementary Abelian p-group.

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1 Introduction and motivation

Let G be any group with a presentation $G \cong F/R$, where F is a free group. Then the Baer invariant of G with respect to the variety of groups $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be

$$\mathcal{V}M(G) = \frac{R \cap V(F)}{[RV^*F]},$$

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where V is the set of words of the variety $\mathcal{V}$, $V(F)$ is the verbal subgroup of F and

$$[RV^*F] = \langle v(f_1, ..., f_{i-1}, f_i r, f_{i+1}, ..., f_n) v(f_1, ..., f_i, ..., f_n)^{-1} | \\ r \in R, f_i \in F, v \in V, 1 \leq i \leq n, n \in \mathbf{N} \rangle.$$

One may check that $\mathcal{V}M(G)$ is abelian and independent of the choice of the free presentation of G. In particular, if $\mathcal{V}$ is the variety of abelian groups, $\mathcal{A}$, then the Baer invariant of the group G will be $(R \cap F')/[R, F]$, which is isomorphic to the well-known notion the Schur multiplier of G, denoted by $M(G)$. If $\mathcal{V}$ is the variety of polynilpotent groups of class row $(c_1, ..., c_s)$, $\mathcal{N}_{c_1,c_2,...,c_s}$, then the Baer invariant of a group G with respect to this variety, which is called a polynilpotent multiplier of G, is as follows:

$$\mathcal{N}_{c_1,c_2,...,c_s}M(G) = \frac{R \cap \gamma_{c_s+1} \circ ... \circ \gamma_{c_1+1}(F)}{[R, \ _{c_1}F, \ _{c_2}\gamma_{c_1+1}(F), ..., \ _{c_s}\gamma_{c_{s-1}+1} \circ ... \circ \gamma_{c_1+1}(F)]},$$

where $\gamma_{c_s+1} \circ ... \circ \gamma_{c_1+1}(F) = \gamma_{c_s+1}(\gamma_{c_{s-1}+1}(...(\gamma_{c_1+1}(F))...))$ are the term of iterated lower central series of F. See Hekster [6] for the equality

$$[R\mathcal{N}^*{}_{c_1,c_2,...,c_s}F] = [R, \ _{c_1}F, \ _{c_2}\gamma_{c_1+1}(F), ..., \ _{c_s}\gamma_{c_{s-1}+1} \circ ... \circ \gamma_{c_1+1}(F)].$$

In particular, if $s = 1$ and $c_1 = c$, then the Baer invariant of G with respect to the variety $\mathcal{N}_c$, which is called the c-nilpotent multiplier of G, is

$$\mathcal{N}_cM(G) \cong \frac{R \cap \gamma_{c+1}(F)}{[R, \ _cF]}.$$

Historically, Green [4] showed that the order of the Schur multiplier of a finite p-group of order p^n is bounded by $p^{\frac{n(n-1)}{2}}$. Berkovich [2] showed that a finite p-group of order p^n is an elementary abelian p-group if and only if the order of $M(G)$ is $p^{n(n-1)/2}$. Moghaddam [15,16] presented a bound for the polynilpotent multiplier of a finite p-group. He showed that if $\mathcal{V}$ is the variety of polynilpotent groups of a given class row and G is a finite d-generator group of order p^n, then

$$|\mathcal{V}M(\mathbf{Z}_p^{(d)})| \leq |\mathcal{V}M(G)||V(G)| \leq |\mathcal{V}M(\mathbf{Z}_p^{(n)})|,$$

where $\mathbf{Z}_n^{(m)}$ denotes the direct sum of m copies of $\mathbf{Z}_n$. As a consequence, using the structure of $\mathcal{V}M(\mathbf{Z}_p^{(n)})$ in [12], we can show that the order of the nilpotent multiplier of a finite p-group of order p^n is bounded by $p^{\chi_{c+1}(n)}$, where $\chi_{c+1}(n)$ is the number of basic commutators of weight $c+1$ on n letters. The first author and Sanati [13] extended a result of Berkovich to the c-nilpotent multiplier of a finite p-group. They showed that for an abelian p-group G, $|\mathcal{N}_cM(G)| = p^{\chi_{c+1}(n)}$ if and only if G is an elementary abelian p-group. Putting an additional condition on the kernel of the left natural map of the generalized Stallings-Stammbach five-term exact sequence, they showed that an arbitrary finite p-group with the c-nilpotent multiplier of maximum order is an elementary abelian p-group.

Unfortunately, there is a mistake in the proof of Theorem 3.5 in [13] due to using the inequality $i\chi_{c+1}(i) < \chi_{c+1}(i+1)$ which is not correct in general. In this paper, first, we give a correct proof for Theorem 3.5 in [13]. Second, we extend the result in different directions. In fact, we show that if $\mathcal{V}$ is the variety of polynilpotent groups of class row $(c_1, c_2, ..., c_s)$, $\mathcal{N}_{c_1,c_2,...,c_s}$, and $G \cong \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$ is the nth nilpotent product of some cyclic p-groups, where $c_1 \geq n$, $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_t$ and $(q,p) = 1$ for all primes q less than or equal to n, then $|\mathcal{N}_{c_1,c_2,...,c_s}M(G)| = p^{d_m}$ if and only if $G \cong \mathbf{Z}_p \overset{n}{*} \mathbf{Z}_p \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_p$ (m-copies), where $m = \sum_{i=1}^{t} \alpha_i$ and $d_m = \chi_{c_s+1}(...(\chi_{c_2+1}(\sum_{j=1}^{n} \chi_{c_1+j}(m)))...)$. Also, we extend the above result to the multiple nilpotent product of cyclic p-groups $G \cong \mathbf{Z}_{p^{\alpha_1}} \overset{n_1}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n_2}{*} ... \overset{n_{t-1}}{*} \mathbf{Z}_{p^{\alpha_t}}$, when $c_1 \geq n_1 \geq ... \geq n_{t-1}$. As a consequence we show that the polynilpotent multiplier of a finite abelian p-group G has maximum order if and only if G is an elementary abelian p-group. Finally we give a similar result for the c-nilpotent multiplier of $G \cong \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$, with the different conditions $n \geq c$ and $(q,p) = 1$ for all primes q less than or equal to $n + c$.

2 Notation and preliminaries

Definition 2.1. Let $\{G_i\}_{i\in I}$ be a family of arbitrary groups. The nth nilpotent product of the family $\{G_i\}_{i\in I}$ is defined as follows:

$$\overset{n}{\underset{i\in I}{\prod^{*}}} G_i = \frac{\prod_{i\in I}^{*} G_i}{\gamma_{n+1}(\prod_{i\in I}^{*} G_i) \cap [G_i]_{i\in I}^{*}},$$

where $\prod_{i\in I}^{*} G_i$ is the free product of the family $\{G_i\}_{i\in I}$, and

$$[G_i]_{i\in I}^{*} = \langle [G_i, G_j] | i, j \in I, i \neq j \rangle^{\prod_{i\in I}^{*} G_i}$$

is the cartesian subgroup of the free product $\prod_{i\in I}^{*} G_i$ which is the kernel of the natural homomorphism from $\prod_{i\in I}^{*} G_i$ to the direct product $\prod_{i\in I}^{\times} G_i$. For further properties of the above notation see Neumann [17]. If $\{G_i\}_{i\in I}$ is a family of cyclic groups, then $\gamma_{n+1}(\prod_{i\in I}^{*} G_i) \subseteq [G_i]^{*}$ and hence $\overset{n}{\prod^{*}_{i\in I}} G_i = \prod_{i\in I}^{*} G_i / \gamma_{n+1}(\prod_{i\in I}^{*} G_i)$.

Definition 2.2. A variety $\mathcal{V}$ is said to be a Schur-Baer variety if for any group G for which the marginal factor group $G/V^{*}(G)$ is finite, then the verbal subgroup $V(G)$ is also finite and $|V(G)|$ divides a power of $|G/V^{*}(G)|$.

Schur proved that the variety of abelian groups, $\mathcal{A}$, is a Schur-Baer variety (see [10]). Also, Baer [1] proved that if u and v have Schur-Baer property, then the variety defined by the word $[u, v]$ has the above property.

The following theorem gives a very important property of Schur-Baer varieties.

Theorem 2.3. (Leedham-Green, McKay [11]). The following conditions on a variety $\mathcal{V}$ are equivalent:

(i) $\mathcal{V}$ is a Schur-Baer variety.
(ii) For every finite group G, its Baer invariant, $\mathcal{V}M(G)$, is of order dividing a power of $|G|$.

In the rest of this section we review some theorems required in the proofs of the main results of the article.

Theorem 2.4. (Jones [9]). Let G be a finite d-generator group of order p^n. Then

$$p^{\frac{1}{2}d(d-1)} \leq |G'||M(G)| \leq p^{\frac{1}{2}n(n-1)}.$$

Theorem 2.5. (Berkovich [2]). Let G be a finite group of order p^n. Then $|M(G)| = p^{\frac{1}{2}n(n-1)}$ if and only if G is an elementary abelian p-group.

Theorem 2.6. (Moghaddam [15,16]). Let $\mathcal{V}$ be the variety of polynilpotent groups of a given class row. Let G be a finite d-generator group of order p^n. Then

$$|\mathcal{V}M(\mathbf{Z}_p^{(d)})| \leq |\mathcal{V}M(G)||V(G)| \leq |\mathcal{V}M(\mathbf{Z}_p^{(n)})|.$$

We recall that the number of basic commutators of weight c on n generators, denoted by $\chi_c(n)$, is determined by Witt formula (see [5]).

Theorem 2.7. (Moghaddam and Mashayekhy [14]). Let $G \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus ... \oplus \mathbf{Z}_{n_k}$ be a finite abelian groups, where n_{i+1} divides n_i for all $1 \leq i \leq k-1$. Then for all $c \geq 1$, the c-nilpotent multiplier of G is

$$\mathcal{N}_cM(G) = \mathbf{Z}_{n_2}^{(b_2)} \oplus \mathbf{Z}_{n_3}^{(b_3-b_2)} \oplus ... \oplus \mathbf{Z}_{n_k}^{(b_k-b_{k-1})},$$

where $b_i = \chi_{c+1}(i)$.

The following result is an interesting consequence of Theorems 2.6 and 2.7.

Corollary 2.8. Let G be a finite d-generator p-group of order p^n, then

$$p^{\chi_{c+1}(d)} \leq |\mathcal{N}_cM(G)||\gamma_{c+1}(G)| \leq p^{\chi_{c+1}(n)}.$$

The following theorem is a generalization of Theorem 2.5.

Theorem 2.9. (Mashayekhy, Sanati [13]). Let G be an abelian group of order p^n. Then $\mathcal{N}_cM(G) = p^{\chi_{c+1}(n)}$ if and only if G is an elementary abelian p-group.

Let $1 \to N \to G \to Q \to 1$ be a $\mathcal{V}$-central extension, where $\mathcal{V}$ is any variety of groups, that is, the above sequence is exact and N is contained in the marginal subgroup of G, $V^*(G)$. Then the following five-term exact sequence exists (see Fröhlich [3]):

$$\mathcal{V}M(G) \xrightarrow{\theta} \mathcal{V}M(Q) \to N \to G/V(G) \to Q/V(Q) \to 1.$$

The nonabelian version of Theorem 2.9 is as follows.

Theorem 2.10. (Mashayekhy, Sanati [13]). Let G be a finite p-group of order p^n. If $\mathcal{N}_cM(G) = p^{\chi_{c+1}(n)}$, then
(i) There is an epimorphism $\mathcal{N}_cM(G) \xrightarrow{\theta} \mathcal{N}_cM(G/G')$ which is obtained from the Fröhlich sequence.
(ii) If $ker(\theta) = 1$, then G is an elementary abelian p-group.

Theorem 2.11. (Mashayekhy, Parvizi [12]). Let

$$G \cong \mathbf{Z}^{(m)} \oplus \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus ... \oplus \mathbf{Z}_{n_k}$$

be a finitely generated abelian group, where n_{i+1} divides n_i for all $1 \leq i \leq k-1$. Then

$$\mathcal{N}_{c_1,c_2,...,c_s}M(G) \cong \mathbf{Z}^{(\beta_m)} \oplus \mathbf{Z}_{n_1}^{(\beta_{m+1}-\beta_m)} \oplus ... \oplus \mathbf{Z}_{n_k}^{(\beta_{m+k}-\beta_{m+k-1})},$$

where $\beta_i = \chi_{c_s+1}(\chi_{c_{s-1}+1}(...(\chi_{c_1+1}(i))...))$ for all $m \leq i \leq m+k$.

Theorems 2.6 and 2.11 imply the following useful inequalities.

Corollary 2.12. With the notation of previous theorem let G be a finite d-generator p-group of order p^n. Then

$$p^{\beta_d} \leq |\mathcal{N}_{c_1,c_2,...,c_s}M(G)||\gamma_{c_s+1}(\gamma_{c_{s-1}+1}(...(\gamma_{c_1+1}(G))...))| \leq p^{\beta_n}.$$

Theorem 2.13. (Hokmabadi, Mashayekhy, Mohammadzadeh [8]). Let $G \cong \underbrace{\mathbf{Z} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}}_{m-copies} \overset{n}{*} \mathbf{Z}_{r_1} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{r_t}$, be the nth nilpotent product of some cyclic groups, where r_{i+1} divides r_i for all $1 \leq i \leq t-1$. If $c \geq n$ and $(p, r_1) = 1$ for all primes p less than or equal to n, then the c-nilpotent multiplier of G is isomorphic to

$$\mathbf{Z}^{(\sum_{i=1}^{n} \chi_{c+i}(m))} \oplus \mathbf{Z}_{r_1}^{(\sum_{i=1}^{n}(\chi_{c+i}(m+1)-\chi_{c+i}(m)))} \oplus ... \oplus \mathbf{Z}_{r_t}^{(\sum_{i=1}^{n}(\chi_{c+i}(m+t)-\chi_{c+i}(m+t-1)))}.$$

Theorem 2.14. (Hokmabadi, Mashayekhy, Mohammadzadeh [8]). Let $G \cong \underbrace{\mathbf{Z} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}}_{m-copies} \overset{n}{*} \mathbf{Z}_{r_1} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{r_t}$ be the nth nilpotent product of some cyclic groups, where r_{i+1} divides r_i for all $1 \leq i \leq t-1$. If $(p, r_1) = 1$ for all primes p less than or equal to n, then the structure of the polynilpotent multiplier of G is

$$\mathcal{N}_{c_1,c_2,...,c_s}M(G) = \mathbf{Z}^{(d_m)} \oplus \mathbf{Z}_{r_1}^{(d_{m+1}-d_m)} \oplus ... \oplus \mathbf{Z}_{r_t}^{(d_{m+t}-d_{m+t-1})},$$

where $d_i = \chi_{c_s+1}(...(\chi_{c_2+1}(\sum_{j=1}^{n} \chi_{c_1+j}(i)))...)$, for all $c_1 \geq n$ and $c_2, ..., c_s \geq 1$ and $m \leq i \leq m+t$.

Theorem 2.15. (Hokmabadi, Mashayekhy [7]). Let $G \cong \underbrace{\mathbf{Z} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}}_{m-copies} \overset{n}{*} \mathbf{Z}_{r_1} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{r_t}$ be the nth nilpotent product of some cyclic groups such that r_{i+1} divides r_i for all $1 \leq i \leq t-1$. If $(p, r_1) = 1$ for any prime p less than or equal to $n+c$, then
(i) if $n \geq c$, then $\mathcal{N}_cM(G) = \mathbf{Z}^{(g_0)} \oplus \mathbf{Z}_{r_1}^{(g_1-g_0)} \oplus ... \oplus \mathbf{Z}_{r_t}^{(g_t-g_{t-1})}$;
(ii) if $c \geq n$, then $\mathcal{N}_cM(G) = \mathbf{Z}^{(f_0)} \oplus \mathbf{Z}_{r_1}^{(f_1-f_0)} \oplus ... \oplus \mathbf{Z}_{r_t}^{(f_t-f_{t-1})}$,
where $f_k = \sum_{i=1}^{n} \chi_{c+i}(m+k)$ and $g_k = \sum_{i=1}^{c} \chi_{n+i}(m+k)$ for $0 \leq k \leq t$.

Theorem 2.16. (Hokmabadi, Mashayekhy, Mohammadzadeh [8]). Let $G \cong A_1 \overset{n_1}{*} A_2 \overset{n_2}{*} \dots \overset{n_k}{*} A_{k+1}$ such that $A_i \cong \mathbf{Z}$ for $1 \le i \le t$ and $A_j \cong \mathbf{Z}_{m_j}$ for $t+1 \le j \le k+1$. Let $c_1 \ge n_1 \ge n_2 \ge \dots \ge n_k$ and $m_{k+1}|m_k|\dots|m_{t+1}$ and $(p, m_{t+1}) = 1$ for all primes $p \le n_1$. Then the structure of the polynilpotent multiplier of G is

$$\mathcal{N}_{c_1,c_2,\dots,c_s}M(G) = \mathbf{Z}^{(e_0)} \oplus \mathbf{Z}_{m_{t+1}}^{(e_t-e_o)} \oplus \dots \oplus \mathbf{Z}_{m_{k+1}}^{(e_k-e_{k-1})},$$

where $e_i = \chi_{c_s+1}(\dots(\chi_{c_2+1}(u + \sum_{j=t}^{i} h_j))\dots)$, for all $t \le i \le k$, $e_0 = \chi_{c_s+1}(\dots(\chi_{c_2+1}(u))\dots)$, $u = \sum_{j=1}^{n_{t-1}} \chi_{c_1+j}(t) + \sum_{i=1}^{t-2} \sum_{j=n_{i+1}+1}^{n_i} \chi_{c_1+j}(i+1)$ and $h_j = \sum_{\lambda=1}^{n_j} (\chi_{c_1+\lambda}(j+1) - \chi_{c_1+\lambda}(j))$.

3 Main results

As we mentioned before, there is a mistake in the proof of Theorem 2.9. More precisely, in the proof of Theorem 3.5 in [13] it is assumed that G is a finite abelian d-generator p-group of order p^n and $|\mathcal{N}_cM(G)| = p^{\chi_{c+1}(n)}$. Then using the inequality $i\chi_{c+1}(i) < \chi_{c+1}(i+1)$ it is proved that $n = d$ and therefore G is an elementary abelian p-group. Unfortunately, the inequality $i\chi_{c+1}(i) < \chi_{c+1}(i+1)$ is not correct and so the proof is not valid.

In this section, first, we intend to present a new proof for Theorem 2.9 in order to remedy the above mentioned mistake. Second, using this new method, we extend the result to polynilpotent multipliers of nilpotent products of cyclic p-groups with some conditions.

Proof of Theorem 2.9.

Proof. Let G be an elementary abelian p-group of order p^n. Then by Theorem 2.7 we have $\mathcal{N}_cM(G) = \mathbf{Z}_p^{(b_2)} \oplus \mathbf{Z}_p^{(b_3-b_2)} \oplus \dots \oplus \mathbf{Z}_p^{(b_n-b_{n-1})}$, where $b_i = \chi_{c+1}(i)$, and hence $|\mathcal{N}_cM(G)| = p^{\chi_{c+1}(n)}$.

Conversely, suppose that $|\mathcal{N}_cM(G)| = p^{\chi_{c+1}(n)}$. Since G is an abelian p-group of order p^n, we can consider G as follows:

$$G \cong \mathbf{Z}_{p^{\alpha_1}} \oplus \mathbf{Z}_{p^{\alpha_2}} \oplus \dots \oplus \mathbf{Z}_{p^{\alpha_d}},$$

where $\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_d$ and $\alpha_1 + \alpha_2 + \dots\alpha_d = n$. By Theorem 2.7 $\mathcal{N}_cM(G) = \mathbf{Z}_{p^{\alpha_2}}^{(b_2)} \oplus \mathbf{Z}_{p^{\alpha_3}}^{(b_3-b_2)} \oplus \dots \oplus \mathbf{Z}_{p^{\alpha_d}}^{(b_d-b_{d-1})}$ and so $|\mathcal{N}_cM(G)| = p^{\alpha_2 b_2 + \alpha_3(b_3-b_2)+\dots+\alpha_d(b_d-b_{d-1})}$. On the other hand by hypothesis $|\mathcal{N}_cM(G)| = p^{b_n}$. Therefore $b_n = \alpha_2 b_2 + \alpha_3(b_3-b_2) + \dots + \alpha_d(b_d-b_{d-1})$. Also $b_n = (b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) + \dots + (b_3 - b_2) + b_2$. Thus

$$(b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) + \dots + (b_3 - b_2) + b_2 = \alpha_2 b_2 + \alpha_3(b_3 - b_2) + \dots + \alpha_d(b_d - b_{d-1}) =$$

$$\underbrace{b_2 + \dots + b_2}_{\alpha_2-copies} + \underbrace{(b_3 - b_2) + \dots + (b_3 - b_2)}_{\alpha_3-copies} + \dots + \underbrace{(b_d - b_{d-1}) + \dots + (b_d - b_{d-1})}_{\alpha_d-copies}.$$

So we have the following equality:

$$(b_n - b_{n-1}) + (b_{n-1} - b_{n-2}) + ... + (b_{d+1} - b_d) =$$

$$\underbrace{b_2 + ... + b_2}_{\alpha_2-1-copies} + \underbrace{(b_3 - b_2) + ... + (b_3 - b_2)}_{\alpha_3-1-copies} + ... + \underbrace{(b_d - b_{d-1}) + ... + (b_d - b_{d-1})}_{\alpha_d-1-copies} \ (I).$$

One can easily see that for any $i \geq 1$, $(b_i - b_{i-1})$ is the number of basic commutators of weight $c+1$ on i letters such that x_i does appear in it. So $(b_j - b_{j-1}) \geq (b_i - b_{i-1})$ whenever $j \geq i$. Now, assume $\alpha_1 \geq 2$. Then $n-1 > n - \alpha_1$ and so the left-hand side of the above equality has more terms than the right-hand side. Also each term of the left-hand side of the above equality is greater than of any term of the right-hand side. These facts imply that the equality (I) dose not hold which is a contradiction. Thus we must have $\alpha_1 = 1$ and hence $d = n$, $\alpha_1 = \alpha_2 = ... = \alpha_n = 1$. Therefore the result holds. □

The next theorem is a generalization of Theorem 2.9. Note that the nilpotent product of finitely many finite p-groups is also a finite p-group.

Theorem 3.1. Let $G \cong \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$ be the nth nilpotent product of some cyclic groups, where $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_t$ and $(q,p) = 1$ for all primes q less than or equal to n. Let $\mathcal{N}_{c_1,c_2,...,c_s}$ be a variety of polynilpotent groups such that $c_1 \geq n$. Then $|\mathcal{N}_{c_1,c_2,...,c_s}M(G)| = p^{d_m}$ if and only if $G \cong \underbrace{\mathbf{Z}_p \overset{n}{*} \mathbf{Z}_p \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_p}_{m-copies}$, where $m = \sum_{i=1}^{t} \alpha_i$ and $d_m = \chi_{c_s+1}(...(\chi_{c_2+1}(\sum_{j=1}^{n} \chi_{c_1+j}(m)))...)$.

Proof. Let $G = \underbrace{\mathbf{Z}_p \overset{n}{*} \mathbf{Z}_p \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_p}_{m-copies}$ and $(q,p) = 1$ for all primes q less than or equal to n. Then by Theorem 2.14,

$$\mathcal{N}_{c_1,c_2,...,c_s}M(G) = \mathbf{Z}_p^{(d_2)} \oplus ... \oplus \mathbf{Z}_p^{(d_m - d_{m-1})},$$

where $d_i = \chi_{c_s+1}(...(\chi_{c_2+1}(\sum_{j=1}^{n} \chi_{c_1+j}(i)))...)$, for all $c_1 \geq n$. Hence

$$|\mathcal{N}_{c_1,c_2,...,c_s}M(G)| = p^{d_m}.$$

Conversely, suppose that $|\mathcal{N}_{c_1,c_2,...,c_s}M(G)| = p^{d_m}$. By the hypothesis $G = \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$ where $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_t$ and $\alpha_1 + \alpha_2 + ... + \alpha_t = m$. Now Theorem 2.14 implies that

$$\mathcal{N}_{c_1,c_2,...,c_s}M(G) = \mathbf{Z}_{p^{\alpha_2}}^{(d_2)} \oplus \mathbf{Z}_{p^{\alpha_3}}^{(d_3-d_2)} ... \oplus \mathbf{Z}_{p^{\alpha_t}}^{(d_t-d_{t-1})},$$

where $d_i = \chi_{c_s+1}(...(\chi_{c_2+1}(\sum_{j=1}^{n} \chi_{c_1+j}(i)))...)$. Thus

$$|\mathcal{N}_{c_1,c_2,...,c_s}M(G) = p^{\alpha_2 d_2 + \alpha_3(d_3-d_2) + ... + \alpha_t(d_t - d_{t-1})}.$$

On the other hand by hypothesis $|\mathcal{N}_{c_1,c_2,...,c_s}M(G)| = p^{d_m}$. Therefore $d_m = \alpha_2 d_2 + \alpha_3(d_3 - d_2) + ... + \alpha_t(d_t - d_{t-1})$. Now applying a similar method to the proof of Theorem 2.9, it is enough to show that if $j \geq i$, then $(d_j - d_{j-1}) \geq (d_i - d_{i-1})$. In order to prove this fact consider the following sets:

$$A_1 = \{\alpha | \alpha \textit{ is a basic commutator of weight } c_1+1, ..., c_1+n \textit{ on } x_1, ..., x_i\}$$

and inductively for all $2 \leq k \leq s$

$$A_k = \{\alpha | \alpha \textit{ is a basic commutator of weight } c_k+1 \textit{ on } A_{k-1}\}.$$

Clearly $d_i = |A_s|$. It is easy to see that

$$d_i - d_{i-1} = |\{\alpha | \alpha \textit{ is a basic commutator of weight } c_s+1 \textit{ on } A_{s-1} \textit{ such that } x_i \textit{ does appear in } \alpha\}|.$$

Hence the required inequality holds. □

Using Theorem 2.16 and a similar proof to the above and noting that $j \geq i$ implies $e_j - e_{j-1} \geq e_i - e_{i-1}$, we can state the following theorem.

Theorem 3.2. Let $G \cong \mathbf{Z}_{p^{\alpha_1}} \overset{n_1}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n_2}{*} ... \overset{n_{t-1}}{*} \mathbf{Z}_{p^{\alpha_t}}$ be the nth nilpotent product of some cyclic groups, where $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_t$ and $(q,p) = 1$ for all primes q less than or equal to n. Let $\mathcal{N}_{c_1,c_2,...,c_s}$ be a variety of polynilpotent groups such that $c_1 \geq n_1$. Then $|\mathcal{N}_{c_1,c_2,...,c_s}M(G)| = p^{e_{m-1}}$ if and only if $G \cong \underbrace{\mathbf{Z}_p \overset{n_1}{*} \mathbf{Z}_p \overset{n_2}{*} ... \overset{n_{m-1}}{*} \mathbf{Z}_p}_{m-copies}$, where $m = \sum_{i=1}^{t} \alpha_i$, $e_{m-1} = \chi_{c_s+1}(...(\chi_{c_2+1}(\sum_{j=0}^{m-1} h_j))...)$, and $h_j = \sum_{\lambda=1}^{n_j}(\chi_{c_1+\lambda}(j+1) - \chi_{c_1+\lambda}(j))$.

The following result is a consequence of Theorem 2.15 and the above mentioned method with different condition $n \geq c$.

Theorem 3.3. Let $G \cong \mathbf{Z}_{p^{\alpha_1}} \overset{n}{*} \mathbf{Z}_{p^{\alpha_2}} \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_{p^{\alpha_t}}$ be the nth nilpotent product of some cyclic groups, where $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_t$, $n \geq c$ and $(q,p) = 1$ for all primes q less than or equal to $n+c$. Then $\mathcal{N}_c M(G) = p^{g_m}$ if and only if $G \cong \underbrace{\mathbf{Z}_p \overset{n}{*} \mathbf{Z}_p \overset{n}{*} ... \overset{n}{*} \mathbf{Z}_p}_{m-copies}$, where $m = \sum_{i=1}^{t} \alpha_i$ and $g_m = \sum_{i=1}^{c} \chi_{n+i}(m)$.

With the assumption and notation of Theorem 3.1, let $n = 1$. Then the nth nilpotent product of $\mathbf{Z}_{p^{\alpha_i}}$ $(1 \leq i \leq t)$ is the direct product of $\mathbf{Z}_{p^{\alpha_i}}$. So G is a finite abelian p-group of order p^m. Also d_i will be equal to β_i in Theorem 2.12. Therefore the following corollary is a consequence of Theorem 3.1.

Corollary 3.4. Let G be an abelian group of order p^m. Then $|\mathcal{N}_{c_1,c_2,...,c_s}M(G)| = p^{\beta_m}$ if and only if G is an elementary abelian p-group, where

$$\beta_m = \chi_{c_s+1}(...(\chi_{c_2+1}(\chi_{c_1+1}(m)))...).$$

Note that according to Corollary 2.12 the polynilpotent multiplier of G in the above result has its maximum order. So the above corollary is a vast generalization of Theorem 2.5.

Finally in order to deal with a non-abelian case we present the following theorem. This theorem is a generalization of Theorem 2.10.

Theorem 3.5. With the previous notation let G be a finite p-group of order p^m. If $\mathcal{N}_{c_1,c_2,\ldots,c_s}M(G)=p^{\beta_m}$, then the following statements holds.
(i) There is an epimorphism $\mathcal{N}_{c_1,c_2,\ldots,c_s}M(G)\xrightarrow{\theta}\mathcal{N}_{c_1,c_2,\ldots,c_s}M(G/G')$ which is obtained from the Fröhlich sequence.
(ii) If $ker(\theta)=1$, then G is an elementary abelian p-group.

Proof. (i) Let $\mathcal{V}$ be the variety of polynilpotent groups of class row $(c_1,c_2,...,c_s)$, $\mathcal{N}_{c_1,c_2,\ldots,c_s}$. By Corollary 2.12 we have $|\mathcal{V}M(G)||V(G)|\le p^{\beta_m}$,. Also by the hypothesis $|\mathcal{N}_{c_1,c_2,\ldots,c_s}M(G)|=p^{\beta_m}$. Therefore $|V(G)|=1$. Now set $N=G'$ and consider the exact sequence $1\to G'\to G\to G/G'\to 1$. Since $|V(G)|=1$, the above sequence is an $\mathcal{V}$-central extension. Therefore by Fröhlich five-term exact sequence we have the following exact sequence:

$$\mathcal{V}M(G)\xrightarrow{\theta}\mathcal{V}M(G/G')\xrightarrow{\beta}G'\xrightarrow{\alpha}G\to G/G'\to 1.$$

Clearly α is a monomorphism and so $Im(\beta)=1$. This means that θ is an epimorphism.
(ii) Let $ker(\theta)=1$. Then $|\mathcal{V}M(G/G')|=|\mathcal{V}M(G)|=p^{\beta_m}\quad(*)$. Since $|G|=p^m$ then $|G/G'|\le p^m$. Hence $|G/G'|=p^m$, otherwise, if $|G/G'|=p^k$, where $k<m$, then $|\mathcal{V}M(G/G')|\le|\mathcal{V}M(G/G')||V(G/G')|\le p^{\beta_k}<p^{\beta_m}$, which is a contradiction to $(*)$. Hence $|G/G'|=p^m$ which implies that G is an abelian p-group. Now by Corollary 3.4 the result holds. □

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