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Numerical Scheme to Solve Integro-Differential Equations System

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Abstract. In this paper, we present a method for numerical solution of linear and nonlinear integro-differential equations system. This method, gives solution of the system by power series and reproduces the analytical solution if the exact solutions are polynomial, otherwise it reproduces their's Taylor series. Comparison of the approximate solution with exact solution shows that the used method is easy and practical for classes of linear and nonlinear integro-differential equations system. The package Maple 9 is used for computation.

Keywords: Integro-differential equation, Power series, Approximate solution.

AMS Subject Classifications: 47G20, 65L05, 33F05.

1 Introduction

There are many physical processes which integro-differential equations arise them, such as nano-hydrodynamics [1], glass-forming process [2], drop wise condensation [3], and wind ripple in the desert [4]. There are several numerical methods for solving system of linear and nonlinear integro-differential equations, for example, the Adomian decomposition methods [5], homotopy perturbation method [6] and [7], Galerkin method [8] and variational iteration method [9]. Recently, the Power series method (PSM) has used for solving stiff ordinary differential equations system [10], [11], linear Volterra integral equations system of the second kind and system of integro-differential equations, [12], [13]. In this paper, we use the Power series method in which the Taylor expansion of the exact solution of linear or nonlinear integro-differential equations system is obtained by recursive procedure.

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To illustration the basic of this method, we consider following integro-differential equations system

$$\boldsymbol{U}'(x) = \boldsymbol{G}(x, \boldsymbol{U}(x)) + \int_0^x \boldsymbol{K}(x, t, \boldsymbol{U}(t), \boldsymbol{U}'(t))dt, \tag{1.1}$$

along with initial condition $\boldsymbol{F}(0) = \boldsymbol{a}$, where

$$\begin{aligned} &\boldsymbol{U} = [u_1, u_2, \dots, u_n]^T, \\ &\boldsymbol{G} = [g_1, g_2, \dots, g_n]^T, \\ &\boldsymbol{K} = [k_{ij}], \quad i, j = 1, 2, \dots, n, \\ &\boldsymbol{a} = [a_1, a_2, \dots, a_n]^T. \end{aligned}$$

In equation (1.1), $\boldsymbol{G}$ and $\boldsymbol{K}$ are given analytic functions and without any loss of generality, we assume that they are polynomials, otherwise they can be substituted by their's Taylor expansion. Also, $\boldsymbol{a}$ is fixed constant vector and the vector function $\boldsymbol{U}$ is the solution of equation (1.1), which will be determined.

2 Power series method

Suppose the solution of the system of integro-differential equations (1.1) be as follow

$$u_i(x) = \sum_{j=0}^{m} e_{ij} x^j, \quad i = 1, 2, \dots, n. \tag{2.1}$$

By using initial conditions, we have

$$e_{i0} = u_i(0), \quad i = 1, 2, \dots, n.$$

We compute coefficients of (2.1) step by step. So, we consider the solution of problem (1.1) as

$$\boldsymbol{U}(x) = \boldsymbol{e}_0 + \boldsymbol{e}_1 x, \tag{2.2}$$

where $\boldsymbol{e}_j = (e_{ij}), i = 1, 2, \dots, n$ and $\boldsymbol{e}_1$ is unknown. By substituting (2.2) into (1.1), we obtain the following system

$$(A_1 \boldsymbol{e}_1 - b_1) + Q_1(x) = 0,$$

where A_1 is $n \times n$ constant matrix, b_1 is $n \times 1$ constant vector, $Q_1(x) = [q_{i1}(x)], i = 1, 2, \dots, n$ and $q_{i1}(x)$ are polynomials of order equal or greater than 1. By neglecting $Q_1(x)$, we have an algebraic linear equations system of $\boldsymbol{e}_1$. By solving this system, the coefficient of x in (2.2) can be determined.

For next step, we assume that

$$\boldsymbol{U}(x) = \boldsymbol{e}_0 + \boldsymbol{e}_1 x + \boldsymbol{e}_2 x^2, \tag{2.3}$$

where $\boldsymbol{e}_0$ and $\boldsymbol{e}_1$ are known and $\boldsymbol{e}_2$ is unknown. By substituting (2.3) into (1.1), we derive the following system

$$(A_2\boldsymbol{e}_2 - b_2)x + Q_2(x) = 0,$$

where A_2 and b_2 are similar to A_1 and b_1, respectively, $Q_2(x) = [q_{i2}(x)], i = 1, 2, \dots, n$ and $q_{i2}(x)$ are polynomials of order greater than unity. By neglecting $Q_2(x)$, we have again an algebraic system of linear equations of $\boldsymbol{e}_2$ and by solving this system, coefficients of x^2 in (2.3) can be determined. This procedure can be repeated till the arbitrary order coefficients of Power series of the solution for the problem be obtained.

The following theorem shows convergence of the method. Without loss of generality, we prove it for $n = 1$.

Theorem 2.1. Let $u = f(x)$ be the exact solution of the following integro-differential equation,

$$u'(x) = g(x, u(x)) + \int_0^x K(x, t, u(x), u'(x))dt, \qquad u(0) = a. \tag{2.4}$$

Furthermore, assume that $f(x)$ has a power series representation. Then, the proposed method obtains it (the Taylor expansion of $f(x)$).

Proof. According to the proposed method, we assume that the approximate solution to Eq.(2.4) be as follows,

$$\widetilde{f}(x) = e_0 + e_1x + e_2x^2 + \cdots . \tag{2.5}$$

Hence, it is sufficient that we only prove,

$$e_m = \frac{f^{(m)}(0)}{m!}, \qquad m = 1, 2, 3, \dots . \tag{2.6}$$

Note that for $m = 0$, the initial condition gives,

$$e_0 = f(0) = a. \tag{2.7}$$

Moreover, for $m = 1$, if we set $u = f(x)$ and $x = 0$ in Eq.(2.4), we obtain

$$f'(0) = g(0, f(0)) + 0. \tag{2.8}$$

On the other hand, from (2.5) and (2.7), we have

$$\widetilde{f}(x) = e_0 + e_1x, \tag{2.9}$$

by substituting (2.9) into Eq.(2.4) and setting $s = 0$, we get

$$e_1 = g(0, f(0)) + 0 = f'(0). \tag{2.10}$$

For $m = 2$, differentiating Eq.(2.4) with respect to x, we have

$$f''(x) = \frac{\partial}{\partial x}g(x, f(x)) + \frac{\partial}{\partial u}g(x, f(x))f'(x) + K(x, f(x), f'(x)) + \int_0^x \frac{\partial}{\partial x}K(x, t, f(t), f'(t)), \tag{2.11}$$

setting $s = 0$ in (2.11) and we get

$$f''(0) = \frac{\partial}{\partial x}g(0, f(0)) + \frac{\partial}{\partial u}g(0, f(0))f'(0) + K(0, f(0), f'(0)). \tag{2.12}$$

According to (2.5),(2.7) and (2.10), let

$$\tilde{f}(x) = f(0) + f'(0)x + e_2x^2, \tag{2.13}$$

by substituting (2.13) into (2.11), and setting $x = 0$, we obtain

$$2e_2 = \frac{\partial}{\partial x}g(0, e_0)) + \frac{\partial}{\partial u}g(0, e_0))e_1 + K(0, e_0, e_1). \tag{2.14}$$

So, with comparison (2.12) and (2.14), we conclude that

$$2e_2 = f''(0), \qquad \text{or} \qquad e_2 = \tfrac{f''(0)}{2!}.$$

By continuing the above procedure, we can easily prove (2.6) for $m = 3, 4, \ldots$. □

Corollary 2.2. If the exact solution to Eq.(2.4) be an polynomial, then the proposed method will be obtained the real solution.

3 Applications

To illustrate the method, we consider three examples of integro-differential equations and then we will compare the obtained results with the exact solutions or the other methods.

Example 3.1. Consider the following system of linear Volterra integro-differential equations [14],

$$\begin{cases} u_1'(x) = 1 + x + x^2 - u_2(x) - \displaystyle\int_0^x (u_1(s) + u_2(s))ds, \\ u_2'(x) = -1 - x + u_1(x) - \displaystyle\int_0^x (u_1(s) - u_2(s))ds, \\ u_1(0) = 1, \quad u_2(0) = -1. \end{cases} \tag{3.1}$$

This problem has the exact solution $u_1^*(x) = x + e^x$, $u_2^*(x) = x - e^x$. From the initial conditions, $\boldsymbol{e}_0 = [1, -1]^T$. Let the solution of (3.1) be

$$\begin{cases} u_1(x) = e_{10} + e_{11}x = 1 + e_{11}x, \\ u_2(x) = e_{20} + e_{21}x = -1 + e_{21}x. \end{cases} \tag{3.2}$$

For obtaining e_{11}, e_{21}, we substitute (3.2) into (3.1) then we will have

$$\begin{cases} (e_{11}-2) + \overbrace{(-x - x^2 + e_{21}x + \frac{1}{2}e_{11}x^2 + \frac{1}{2}e_{21}x^2)}^{q_{11}(x)} = 0, \\ e_{21} + \underbrace{(3x - e_{11}x + \frac{1}{2}e_{11}x^2 - \frac{1}{2}e_{21}x^2)}_{q_{21}(x)} = 0, \end{cases}$$

where $q_{11}(x), q_{21}(x)$ are $O(x)$ and by neglecting them, we have

$$A_1\boldsymbol{e}_1 = b_1,$$

where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_1 = \begin{bmatrix} e_{11} \\ e_{21} \end{bmatrix}.$$

So,

$$\boldsymbol{e}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$

and then

$$\begin{cases} u_1(x) = 1 + 2x, \\ u_2(x) = -1. \end{cases}$$

We go to next step. Let

$$\begin{cases} u_1(x) = 1 + 2x + e_{12}x^2, \\ u_2(x) = -1 + e_{22}x^2. \end{cases} \tag{3.3}$$

Similar to previous step, by substitute (3.3) into (3.1), we have

$$\begin{cases} (2e_{11}-1)x + \overbrace{(e_{22}x^2 + \frac{1}{3}e_{12}x^3 + \frac{1}{3}e_{22}x^3)}^{q_{12}(x)} = 0, \\ (2e_{22}+1)x + \underbrace{(-e_{12}x^2 + \frac{1}{3}e_{12}x^3 - \frac{1}{3}e_{22}x^3 + x^2)}_{q_{22}(x)} = 0. \end{cases}$$

By neglecting $q_{12}(x), q_{22}(x)$ which are $O(x^2)$ and solve system $A_2\boldsymbol{e}_2 = b_2$, we obtain $\boldsymbol{e}_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$ and then

$$\begin{cases} u_1(x) = 1 + 2x + \frac{1}{2}x^2, \\ u_2(x) = -1 - \frac{1}{2}x^2. \end{cases}$$

Summery of next step is

$$\begin{cases} u_1(x) = 1 + 2x + \dfrac{1}{2}x^2 + e_{13}x^3, \\ u_2(x) = -1 - \dfrac{1}{2}x^2 + e_{23}x^3, \end{cases}$$

and

$$\begin{cases} (3e_{13} - \dfrac{1}{2})x^2 + \overbrace{(e_{23}x^3 + \dfrac{1}{4}e_{13}x^4 + \dfrac{1}{4}e_{23}x^4)}^{q_{13}(x)} = 0, \\ (3e_{23} + \dfrac{1}{2})x^2 + \underbrace{(-e_{13}x^3 + \dfrac{1}{4}e_{13}x^4 - \dfrac{1}{4}e_{23}x^4 + 1/3x^3)}_{q_{23}(x)} = 0. \end{cases}$$

So $\boldsymbol{e}_3 = \begin{bmatrix} \frac{1}{6} \\ -\frac{1}{6} \end{bmatrix}$ and then

$$\begin{cases} u_1(x) = 1 + 2x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3, \\ u_2(x) = -1 - \dfrac{1}{2}x^2 - \dfrac{1}{6}x^3. \end{cases}$$

The rest of components of the solution by iteration method can be obtained in a similar way.

$$\begin{cases} u_1(x) = 1 + 2x + \dfrac{1}{2}x^2 + \dfrac{1}{6}x^3 + \dfrac{1}{24}x^4 + \dfrac{1}{120}x^5 + \dfrac{1}{720}x^6, \\ u_2(x) = -1 - \dfrac{1}{2}x^2 - \dfrac{1}{6}x^3 - \dfrac{1}{24}x^4 - \dfrac{1}{120}x^5 - \dfrac{1}{720}x^6. \end{cases}$$

One can see that these solutions tend to the exact solutions, in the other word, the exact solutions are

$$\begin{cases} u_1^*(x) = u_1(x) + O(x^7), \\ u_2^*(x) = u_2(x) + O(x^7). \end{cases}$$

where $O(x^7)$ is the reminder, which define the error between the exact solution and the Taylor polynomial solution.

Example 3.2. As second example, we consider the following nonlinear integro-differential equation [7],

$$\begin{cases} u'(x) = 1 + \displaystyle\int_0^x u(s)u'(s)ds, \\ u(0) = 0. \end{cases} \tag{3.4}$$

Typically, we use the Power series method for obtaining the solution of the problem. From the initial condition, $e_0 = 0$. Let the solution of (3.4) is the form

$$u(x) = e_0 + e_1x = e_1x. \tag{3.5}$$

For obtaining e_1, we substitute (3.5) into (3.4), we will have

$$(e_1 - 1) + \overbrace{(-\frac{1}{2}e_1^2 x^2)}^{q_1(x)} = 0.$$

By neglecting $q_1(x)$ which is $O(x)$, we obtain $e_1 = 1$ and then $u(x) = x$. For the next step, we assume

$$u(x) = x + e_2 x^2. \tag{3.6}$$

By substituting (3.6) into (3.4), we have

$$2e_2 x + \overbrace{(-\frac{1}{2}e_2^2 x^4 - e_2 x^3 - \frac{1}{2}x^2)}^{q_2(x)} = 0.$$

From above relation and by neglecting $q_2(x)$, we have $e_2 = 0$ and $u(x)$ is the same as in previous step. By repeating this method, we can compute more coefficients of the solution. We have computed these coefficients till e_{11} and the result is

$$u(x) = x + \frac{1}{6}x^3 + \frac{1}{30}x^5 + \frac{17}{2520}x^7 + \frac{31}{22680}x^9 + \frac{691}{2494800}x^{11}.$$

It is easy to verify that the exact solution of (8) is

$$u^*(x) = \sqrt{2}\tan(\frac{\sqrt{2}}{2}x) = u(x) + O(x^{12}),$$

where $O(x^{12})$ is the reminder of the Taylor polynomial solution.

The results and the corresponding absolute errors are presented in Table 1. Last column of Table 1, is absolute error of homotopy perturbation method (HPM) [7]. We can see the results of the Power series method is better than homotopy perturbation method with same terms of approximate solution.

Table 1. Numerical results of Example 3.2

x	$Exactsolution$	$PowerSeriesMethod$	$Error(PSM)$	$Error(HPM)$
0	0	0	0	0
0.06250	0.06254072191846	0.06254072191846	0	0
0.12500	0.12532654131291	0.12532654131291	0	0
0.18750	0.18860641293011	0.18860641293009	0	0
0.25000	0.25263713577608	0.25263713577523	0	0
0.31250	0.31768760708140	0.31768760706588	0	0
0.37500	0.38404349626063	0.38404349609309	0	1×10^{-5}
0.43750	0.45201251729751	0.45201251604129	0	2×10^{-5}
0.50000	0.52193051544762	0.52193050823056	0	5×10^{-5}
0.56250	0.59416863732287	0.59416860347407	0	1.2×10^{-4}
0.62500	0.66914192777183	0.66914179243171	1.35×10^{-7}	2.4×10^{-4}
0.68750	0.74731980057855	0.74731932475829	4.75×10^{-7}	4.7×10^{-4}
0.75000	0.82923897486139	0.82923746989145	1.50×10^{-6}	8.6×10^{-4}
0.81250	0.91551967310620	0.91551531552761	4.36×10^{-6}	1.51×10^{-3}
0.87500	1.00688616720544	1.00687445884978	1.17×10^{-5}	2.56×10^{-3}
0.93750	1.10419317751013	1.10416366668469	2.95×10^{-5}	4.17×10^{-3}
1.00000	1.20846024213727	1.20838985088985	7.03×10^{-5}	6.61×10^{-3}

Example 3.3. Consider the following nonlinear integro-differential equation,

$$\begin{cases} u'(x) = e^x - \frac{1}{3}e^{3x} + \frac{1}{3} + \int_0^x u^3(s)ds, \\ u0) = 1, \end{cases} \tag{3.7}$$

with the exact solution $u^*(x) = e^x$. In this example, in mth step, we use $m+1$ terms of Taylor expansion of e^x and e^{3x} . Again, we use the Power series method for obtaining the solution of the problem. From the initial condition, $e_0 = 1$. Assume, the solution of (3.7) is the form

$$u(x) = e_0 + e_1 x = 1 + e_1 x. \tag{3.8}$$

By substituting (3.8) into (3.7), we obtain,

$$(e_1 - 1) + \overbrace{(-x - \frac{3}{2}e_1 x^2 - e_1^2 x^3 - \frac{1}{4}e_1^3 x^4)}^{q_1(x)} = 0.$$

By neglecting $q_1(x)$, we obtain $e_1 = 1$ and then $u(x) = 1 + x$. For the next step, we assume

$$u(x) = 1 + x + e_2 x^2, \tag{3.9}$$

and by substituting it into (3.7), we have

$$(2e_2 - 1)x + \overbrace{(-\frac{1}{2}x^2 - \frac{1}{74}e_2^3 x^7 - \frac{1}{2}e_2^2 x^6 - \frac{3}{5}e_2^2 x^5 - \frac{3}{5}e_2 x^5 - \frac{3}{2}e_2 x^4 - \frac{1}{4}x^4 - e_2 x^3 - x^3)}^{q_2(x)} = 0.$$

From the above relation and by neglecting $q_2(x)$, we have $e_2 = \frac{1}{2}$ and we obtain $u(x) = 1 + x + \frac{1}{2}x^2$. By continuing this procedure, more coefficients of the solution can be computed. These coefficients till e_7 have been computed and the result is

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \frac{1}{5040}x^7.$$

This function is exactly the first eight terms of Taylor expansion of the analytic solution. So,

$$u^*(x) = u(x) + O(x^8).$$

and $O(x^8)$ is the error between the exact solution and the Taylor polynomial expansion.

Example 3.4. Finally, consider the following nonlinear integro-differential equation [15],

$$\begin{cases} x''(t) + 2tx'^2(t) = \displaystyle\int_0^t (ts^2 \ln x'^2(s) + y''(s))ds, \\ y''(t) = 2 - t^2 + \displaystyle\int_0^t (\ln(y''(s) - x'(s)) + x'(s)y'(s))ds, \\ x(0) = 0, x'(0) = 1, \\ y(0) = y'(0) = 0. \end{cases} \tag{3.10}$$

This system has the exact solution $x^*(t) = t$ and $y^*(t) = t^2$. Let

$$u_1(t) = x(t),\ u_2(t) = x'(t),\ u_3(t) = y(t),\ u_4(t) = y'(t),$$

so, the system (3.10) convert to

$$\begin{cases} u_1'(t) = u_2(t), \\ u_2'(t) + 2tu_2^2(t) = \displaystyle\int_0^t (ts^2 \ln u_2^2(s) + u_4'(s))ds, \\ u_3'(t) = u_4(t), \\ u_4'(t) = 2 - t^2 + \displaystyle\int_0^t (\ln(u_4'(s) - u_2(s)) + u_2(s)u_4(s))ds, \\ u_1(0) = 0, u_2(0) = 1, u_3(0) = 0, u_4(0) = 0. \end{cases} \tag{3.11}$$

Let $\boldsymbol{U}(t) = [u_1(t), u_2(t), u_3(t), u_4(t)]^T$ and $\boldsymbol{e}_i = [e_{1i}, e_{2i}, e_{3i}, e_{4i}]^T$. From the initial condition, $\boldsymbol{e}_0 = [0, 1, 0, 0]^T$ and so, $\boldsymbol{U}(t) = \boldsymbol{e}_0$. Assume, the solution of (3.11) is the form

$$\boldsymbol{U}(t) = \boldsymbol{e}_0 + t\boldsymbol{e}_1 = [e_{11}t,\ 1 + e_{21}t,\ e_{31}t,\ e_{41}t]^T \tag{3.12}$$

By using the Taylor series of $\ln u_2$ about 1 and $\ln(u_4'(s) - u_2(s))$ about $e_{41} - 1$ and substituting (3.12) into (3.11), we obtain,

$$\begin{cases} (e_{11}-1)+\overbrace{(-e_{21}t)}^{q_{11}(t)}=0, \\ e_{31}+\overbrace{(-e_{41}t)}^{q_{21}(t)}=0, \\ e_{21}+\overbrace{[(2-e_{41})t+4e_{21}t^2+2e_{21}^2t^3-\frac{2}{3}e_{21}t^5+\frac{1}{3}e_{21}^2t^6-\frac{2}{9}e_{21}^3t^7+\frac{1}{9}e_{21}^4t^8]}^{q_{31}(t)}=0, \\ (e_{41}-2)+\overbrace{[(1+\frac{1}{2}e_{21}-\frac{1}{2}e_{41})t^2+(\frac{1}{6}e_{21}^2-\frac{1}{3}e_{21}e_{41})t^3+\frac{1}{12}e_{21}^3t^4+\frac{1}{20}e_{21}^4t^5]}^{q_{41}(t)}=0, \end{cases}$$

By neglecting $q_{i1}(t), i=1,2,3,4$, we obtain the following

$$\boldsymbol{U}(t)=[t,\ 1,\ 0,\ 2t]^T. \tag{3.13}$$

For the next step, we assume

$$\boldsymbol{U}(t)=[t,\ 1,\ 0,\ 2t]^T+\boldsymbol{e}_2t^2, \tag{3.14}$$

and by substituting it into (3.11), we have

$$\begin{cases} 2e_{12}t+O(t^2)=0, \\ (2e_{32}-2)t+O(t^2)=0, \\ 2e_{22}+O(t^2)=0, \\ 2e_{42}+O(t^2)=0, \end{cases}$$

From the above relation , we have $\boldsymbol{e}_2=[0,\ 0,\ 1,\ 0]^T$ and we obtain $\boldsymbol{U}(t)=[t,\ 1,\ t^2,\ 2t]^T$. By continuing this procedure, we will obtain zero for all of the remain coefficients. So, for this example, we have been computed the exact solution.

4 Conclusions

In this work, we have used Power series method for numerical solution of linear and nonlinear Volterra Integro-differential equations system. As shown in the three examples of this paper, the proposed method is a powerful procedure for solving the problems. The simplicity and also easy-to-apply in programming are two special features of this method.

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