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A NOTE ON ANALYTICAL OF DOUBLE LINEAR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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abstract. *In this paper we investigate the VOLTERRA system of double linear integral equation of the second kind with difference kernels. We point out the condition for the existence of a unique solution and finally present some illustrations.*

1 Introduction

In this section,we introduce some concepts required to develop this article. By a double Carson-Laplace integral transformation (DCL) of bounded integrable function $f(x,y)$, we mean

$$F(p,q) = pq\int_0^\infty\int_0^\infty e^{(-px-qy)} f(x,y)dxdy\,, \qquad (1)$$

We denote it as follows:

$$F(p,q) \doteqdot f(x,y)\,.$$

Also, a double convolution of two bounded INTEGRABLE functions is defined as

$$f_1(x,y) \underset{**}{\overset{xy}{}} f_2(x,y) = \int_0^\infty\int_0^\infty f_1(x-\eta, y-\xi) f_2(\eta,\xi) d\eta d\xi\,.$$

1.1 Double Convolution Theorem (DCT)

If $F_1(p,q) \doteqdot f_1(x,y), F_2(p,q) \doteqdot f_2(x,y), F(p,q) = \frac{1}{pq} F_1(p,q) F_2(p,q)$ and suppose $f(x,y) = \int_0^x \int_o^y f_1(x-\eta, y-\xi) f_2(\eta,\xi) d\eta d\xi$, then $F(p,q) \doteqdot f(x,y)$.

For the proof of this theorem see [3].

2 The system of double linear VOLTERRA integrable equations of the second kind

In the following, we are going to explain the method of the solution of the system double linear VOLTERRA integral equations (SDV) as well as pointing out the condition for the existence and uniqueness for this solution.

2.1 The Method of the Solution

Let us consider the following SDV:

$$\phi_i(x,y) = f_i(x,y) + \sum_{m=1}^{s} \int_0^x \int_0^y k_{im}(x-\eta, y-\xi) \phi_m(\eta,\xi) d\eta d\xi \ , \ i = 1,2,\dots,s . \qquad (2)$$

Where $\phi_i(x,y), i = 1,2,\dots,s$ are unknown functions and $f_i(x,y)$ and $k_{im}(x,y), i = 1,2,\dots,s$ are known functions.

To solve this system of integral equations, we assume that $\Phi_i(p,q), F_i(p,q)$ and $K_i(p,q), i = 1,2,\dots,s$ are double Carson-Laplace integral transformation of $\phi_i(x,y), f_i(x,y)$ and $k_i(x,y)$ respectively.

We apply DCL on both sides of (2) and next we use DCT to obtain the following systems:

$$\Phi_i(p,q) = F_i(p,q) + \frac{1}{pq} \sum_{m=1}^{s} K_{im}(p,q) \Phi_m(p,q) , i = 1,2,\dots,s . \qquad (3)$$

In system (3) $\Phi_m(p,q), m = 1,2,\dots,s$ are unknown functions. To obtain $\Phi_i(p,q), i = 1,2,\dots,s$ we solve the system of bivariate linear equations, to obtain

$$\big(1 - K_{ii}(p,q)\big)\Phi_i(p,q) - \frac{1}{pq} \sum_{m=1}^{s} K_{im}(p,q) \Phi_m(p,q) = F_i(p,q) \quad i = 1,2,\dots,s . \qquad (4)$$

The system (4) has a unique solution if and only if the determinant of the coefficients is not equal to zero, namely

$$\begin{vmatrix} 1 - K_{11}(p,q) & -\frac{1}{pq} K_{12}(p,q) & \dots & -\frac{1}{pq} K_{1s}(p,q) \\ -\frac{1}{pq} K_{21}(,qp) & 1 - K_{22}(p,q) & \dots & -\frac{1}{pq} K_{2s}(p,q) \\ \vdots & & \ddots & \vdots \\ -\frac{1}{pq} K_{s1}(p,q) & -\frac{1}{pq} K_{s2}(p,q) & \dots & 1 - K_{ss}(p,q) \end{vmatrix} \neq 0 \qquad (5)$$

Thus, (5) is the condition for existence and solutions of the equations (4), which are desired. The following example explains the use of the application of this kind of system of equations.

2.2 Example 1 Solve the following system of equations.

$$\begin{cases} \phi_1(x,y) = -12\int_0^x\int_0^y \phi_1(\eta,\xi)d\eta d\xi + \int_0^x\int_0^y \phi_2(\eta,\xi)d\eta d\xi + \begin{cases} x^3+y^2+\frac{x^4y}{2}-\frac{x^5}{10}, & y>x, \\ x^2y^3+\frac{xy^4}{2}-\frac{y^5}{10}, & y<x. \end{cases} , \\ \phi_2(x,y) = 6\int_0^x\int_0^y \phi_1(\eta,\xi)d\eta d\xi + 6\int_0^x\int_0^y (x-\eta)(y-\xi)\phi_2(\eta,\xi)d\eta d\xi + \begin{cases} x-x^3y^2-x^4y^2+\frac{x^5}{10}, & y>x, \\ y-x^2y^3+\frac{xy^4}{2}-\frac{y^5}{10}, & y<x. \end{cases} . \end{cases} \quad (6)$$

By taking DCL of both sides of equations (6), we have

$$\begin{cases} \Phi_1(p,q) = \frac{-12}{pq}\Phi_1(p,q) + \frac{1}{pq}\Phi_2(p,q) + \frac{12}{p^2q^2(p+q)} , \\ \Phi_2(p,q) = \frac{6}{pq}\Phi_1(p,q) + \frac{6}{p^2q^2} + \frac{p^2q^2-12}{p^2q^2(p,q)} . \end{cases} \quad (7)$$

Obviously, the condition from (5) is satisfied,by solving (7) we obtain

$$\Phi_1(p,q) = \frac{1}{pq(p+q)} \quad , \quad \Phi_2(p,q) = \frac{1}{p+q} . \quad (8)$$

Now, by taking DCL of both sides of the above equations, we obtain the solution of (6) as follows:

$$\phi_1(x,y) = \begin{cases} x^2y - \frac{x^3}{3} , & y>x , \\ xy^2 - \frac{y^3}{3} , & y<x . \end{cases} \quad , \quad \phi_2(x,y) = \begin{cases} x , & y>x , \\ y , & y<x . \end{cases} \quad (9)$$

3 The solution of double convolution Volterra integral equations of the second kind (SDC)

In this section we are looking for the solution of an SDC as well as considering the condition for the existence and uniqueness of this solution.

$$\sum_{m=1}^{s}\int_0^x\int_0^y k_{im}(x-\eta,y-\xi)\phi_m(\eta,\xi)d\eta d\xi = f_i(x,y) , \quad i = 1,2,\dots,s. \quad (10)$$

Where $\phi_m(x,y)$, $m = 1,2,\dots,s$ and $f_i(x,y)$, $i = 1,2,\dots,s$ are unknown functions, the solution for (10) can be developed in the same way as we did for the SDV (2). We will again use the double Carson-Laplace integral transformation to get the following system of two variables linear algebraic equations.

$$\frac{1}{pq}\sum_{m=1}^{s} K_{im}\Phi_m(p,q) = F_i(p,q) , \quad i = 1,2,\dots,s. \quad (11)$$

Where and are double Carson-Laplace integral transform of and respectively.

System (11) has a nontrivial unique solution if and only if the following determinant does not vanishes, namely

$$\begin{vmatrix} \frac{1}{pq}K_{11}(p,q) & \frac{1}{pq}K_{12}(p,q) & \dots & \frac{1}{pq}K_{1s}(p,q) \\ \vdots & & \ddots & \vdots \\ \frac{1}{pq}K_{s1}(p,q) & \frac{1}{pq}K_{s2}(p,q) & \dots & \frac{1}{pq}K_{ss}(p,q) \end{vmatrix} \neq 0 \qquad (12)$$

Now, we take the following example to illustrate the application of the method to get the analytic solutions of the system integral equation.

3.1 Example2 Consider the following system of integral equations.

$$\begin{cases} \int_0^x\int_0^y \left(1+J_0\left(2\sqrt{(x-\eta)(y-\xi)}\right)\right)\phi_1(\eta,\xi)d\eta d\xi - \int_0^x\int_0^y \phi_2(\eta,\xi)d\eta d\xi = \int_0^x\int_0^y \eta\xi\, F_{12}(1;2,2;-\eta\xi)d\eta d\xi\ , \\ -\int_0^x\int_0^y \phi_2(\eta,\xi)d\eta d\xi + \int_0^x\int_0^y \left(1+(x-\eta)(y-\xi)\right)\phi_2(\eta,\xi)d\eta d\xi = xy\,. \end{cases} \qquad (13)$$

Doing the same way as we did for example 1 and using formulas on pages 115,413 and 427 of [7],we get

$$\begin{cases} \frac{1}{pq}\Phi_1(p,q)+\frac{1}{pq}\frac{pq}{pq+1}\Phi_1(p,q)-\frac{1}{pq}\Phi_2(p,q)=\frac{1}{pq}\frac{1}{pq+1}, \\ -\frac{1}{pq}\Phi_1(p,q)+\frac{1}{pq}\Phi_2(p,q)+\frac{1}{pq}\frac{1}{pq}\Phi_2(p,q)=\frac{1}{pq}. \end{cases} \qquad (14)$$

Obviously the determinant of the coefficient in (14) is not example to zero namely

$$\begin{vmatrix} \frac{1}{pq}+\frac{1}{pq+1} & -\frac{1}{pq} \\ -\frac{1}{pq} & \frac{1}{pq}+\frac{1}{p^2q^2} \end{vmatrix} \neq 0$$

Therefore, the system (14) has the following unique solution

$$\Phi_1(p,q)=1 \qquad , \qquad \Phi_2(p,q)=\frac{2pq}{1+pq} \qquad (15)$$

Taking the inverse double Carson-Laplace transform by using formula p.98, p.418 [7] we obtain the solution

$$\phi_1(x,y)=1 \qquad , \qquad \phi_2(x,y)=2J_0\left(2\sqrt{xy}\right)$$

3.2 Example3 Determination of the exact solution of the following system of integral equations.

$$\begin{cases} \int_0^x\int_0^y \left(1-ber\left(2\sqrt{(x-\eta)(y-\xi)}\right)\right)\phi_1(\eta,\xi)d\eta d\xi - \int_0^x\int_0^y \phi_2(\eta,\xi)d\eta d\xi = \frac{x^4y^4}{4!\,4!}, \\ \int_0^x\int_0^y (x-\eta)^{-1}(y-\xi)^{-1}F_{14}\left(1;1,\frac{3}{2},1,\frac{3}{2};\frac{(x-\eta)^2(y-\xi)^2}{6}\right)\phi_1(\eta,\xi)d\eta d\xi + \int_0^x\int_0^y \phi_2(\eta,\xi)d\eta d\xi = \frac{x^5y^5}{6!\,6!}. \end{cases} \qquad (16)$$

The system of the transform equation of (16) can be obtained in a similar manner of example 2, using the formula on page 295 of [4] and pages 115,398 and 431 of [7], we get

$$\begin{cases} \frac{1}{pq}\Phi_1(p,q) - \frac{1}{pq}\frac{p^2q^2}{p^2q^2+1}\Phi_1(p,q) - \frac{1}{pq}\Phi_2(p,q) = \frac{1}{p^4q^4} \\ \frac{1}{pq}\frac{3}{p^2q^2+1}\Phi_1(p,q) + \frac{3}{pq}\Phi_2(p,q) = \frac{3}{p^5q^5} \end{cases} \quad (17)$$

It is easy to see that the determinant of the coefficient of the system (17) is not equal to zero, thus we obtain

$$\Phi_1(p,q) = \frac{1}{2}\frac{1}{p^3q^3}(p^2q^2+1) + \frac{1}{2p^4q^4}(p^2q^2+1) \quad (18)$$

And

$$\Phi_2(p,q) = \frac{1}{p^4q^4} - \frac{1}{p^2q^2+1}\Phi_1(p,q) \quad (19)$$

Using formulas p.282 [7], we have

$$\phi_1(x,y) = \frac{1}{2}x^{-5}y^{-5}\ {}_1F_4\left(1;1,\frac{3}{2},1,\frac{3}{2};\frac{x^2y^2}{16}\right) + \frac{1}{8}x^{-6}y^{-6}\ {}_1F_4\left(-1;\frac{3}{2},2,\frac{3}{2},2;\frac{x^4}{16}\right) \quad (20)$$

Substituting (18) in (19), we get

$$\Phi_2(p,q) = \frac{1}{p^4q^4} - \frac{1}{p^2q^2+1}\left[\frac{1}{2p^3q^3}(p^2q^2+1) + \frac{1}{2p^4q^4}(p^2q^2+1)\right]\frac{1}{p^4q^4} - \frac{1}{2p^3q^3} + \frac{1}{2p^4q^4}\frac{3}{2p^4q^4} - \frac{1}{2p^3q^3} \quad (21)$$

Now, using formula p.398 [7] we have $\phi_2(x,y) = \frac{3}{2}\frac{x^4y^4}{4!4!} - \frac{1}{2}\frac{x^3y^3}{3!3!}$.

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