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INVARIANCE OF PRIMITIVE IDEALS BY Φ-DERIVATIONS ON BANACH ALGEBRAS

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Abstract. We show that in certain cases a Φ-derivation on a Banach algebra with a nilpotent separating ideal leaves each primitive ideal invariant. We also obtain some sufficient conditions for the separating ideal of a Φ-derivation to be nilpotent.

1. INTRODUCTION

In this paper we study Φ-derivations on Banach algebras. Following [3] by a Φ-derivation on an algebra A , we mean a linear mapping Δ: $A \to A$ which satisfies

$$\Delta(xy) = \Delta(x)\Phi(y) + x\Delta(y) \qquad (x, y \in A),$$

where Φ is an automorphism on A.

If τ denotes the identity map on A, then τ-derivations would be the ordinary derivations on A. Also for every automorphism Φ on A, τ-Φ is a Φ-derivation, and for each fixed $c \in A$ the mapping $\Delta(x)$=$c\Phi(x) - cx$ ($x \in A$), is a Φ-derivation which is called an inner Φ-derivation. Moreover, if D is an ordinary derivation on A and if b is an invertible element in A, then the map $x \mapsto D(x)b$ is a Φ-derivation on A where Φ is the inner automorphism $x \mapsto b^{-1}xb$.

These objects have been considered extensively in algebraic point of view, see for example [1, 2] and [4]. They also have been used in [2] to study Jordan automorphisms on Banach algebras. Brešar and Villena in [3] obtained some algebraic technical results about Φ-derivations and by applying them they proved some results concerning Φ-derivations of Banach algebras. The following theorem is the final result of [3]. Here $Rad(A)$ denotes the Jacobson radical of A.

Theorem A. *Consider the following assertions.*

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(i) *For every inner automorphism Φ and every Φ-derivation Δ of a unital Banach algebra A, Δ leaves each primitive ideal of A invariant.*

(ii) *For every inner automorphism Φ and every Φ-derivation Δ of a unital Banach algebra A, $\Delta(a)$ is quasinilpotent whenever $a \in Rad(A)$ is such that $\Delta^2(a) = 0$.*

(iii) *For every inner automorphism Φ and every Φ-derivation Δ of a unital Banach algebra $A, \Delta(a) \neq 1$ for every $a \in Rad(A)$.*

(iv) *Every derivation on a Banach algebra A leaves each primitive ideal of A invariant.*

(v) *Every derivation on a unital Banach algebra A takes invertible values only on such elements $a \in A$ for which the two sided ideal of A generated by a equals A.*

Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Leftrightarrow (v)$.

Assertion (iv) is the well known noncommutative Singer-Wermer conjecture.

In section 2 we show that if Δ is a Φ-derivation of a unital Banach algebra with Φ a continuous automorphism, such that both Φ and $[\Delta, \Phi] := \Delta\Phi - \Phi\Delta$ leave each nilpotent and each primitive ideal invariant (e.g. Φ is inner) and if $S(\Delta)$, the separating space of Δ, is nilpotent then Δ leaves each primitive ideal invariant. This is a generalization of [3, Corollary 3.4]. Also we may add a new assertion to Theorem A as follows.

(i') *For every inner automorphism Φ and every Φ-derivation Δ of a unital Banach algebra A, Δ has a nilpotent separating ideal.*

Then $(i') \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Leftrightarrow (v)$.

This naturally leads us to the following questuion.

Q1. *Is it true that each Φ-derivation on a Banach algebra has a nilpotent separating space?*

It is indeed an open problem for ordinary derivations and it is shown in [8] that for ordinary derivations it is equivalent to the noncommutative Singer-Wermer conjecture. In Section 3 we obtain some sufficient conditions for Δ to have a nilpotent separating ideal and hence leave each primitive ideal invariant.

Note that if Φ is an automorphism and if Δ is a Φ-derivation on a nonunital algebra A, then we may extend it to the unitization of A by defining $\Delta(1) = 0$. Throughout this paper A is a unital Banach algebra, Φ is a continuous automorphism on A and Δ is a Φ-derivation of A. For a Banach algebra A, the sets R and B denote the Jacobson radical and the Baer radical of A, respectively. It is clear that

B and R are invariant under each automorphism Φ on A. The separating space, $S(\Delta)$ of Δ is defined to be the set

$$S(\Delta) := \{a \in A : \exists\{a_n\} \subseteq A, a_n \to 0, \Delta(a_n) \to a\};$$

which is a closed subspace of A and by the closed graph theorem $S(\Delta) = \{0\}$ if and only if Δ is continuous. For a moment consider A as a Banach A-bimodule, denoted by A^o, with module operations, $A \times A^o \to A^o$, $(a, x) \mapsto a.x = a\Phi(x)$, $(x, a) \mapsto x.a = xa$, for all $a, x \in A$. Obviously Δ is an intertwining map from A into A^o. Thus by [6, Theorem 5.2.15], $S(\Delta)$ is a separating submodule and hence a separating ideal of A, by surjectivity of Φ.

2. Δ-Invariant Ideals

Cusack in [5] proved that each derivation on a Banach algebra leaves the Baer radical invariant. Here we prove a similar result for Φ-derivations, where Φ is a continuous automorphism and Φ, $[\Delta, \Phi]$ leave each nilpotent ideal invariant. Clearly these conditions hold if Φ is inner.

Theorem 2.1. *Let Δ be a Φ-derivation on A, such that Φ and $[\Delta, \Phi]$ leave each nilpotent ideal invariant. Then $\Delta(B) \subseteq B$.*

Proof. Let I be a nilpotent ideal with $I^k = \{0\}$. Take $a \in I$ and $b_1, b_2, ..., b_k \in A$, then by assumption, $(b_1a)(\Phi^{-1}(b_2a))...(\Phi^{-(k-1)}(b_ka)) = 0$. Hence by [3, Theorem 2.3]

$$0 = \Delta^k((b_1a)(\Phi^{-1}(b_2a))...(\Phi^{-(k-1)}(b_ka))) + I = k!\Delta(b_1a)\Delta(b_2a)...\Delta(b_ka) + I.$$

But $\Delta(b_ia) + I = b_i\Delta(a) + \Delta(b_i)\Phi(a) + I = b_i\Delta(a) + I$ for $i = 1, ...k$. Thus $(A\Delta(a))^k \subseteq I \subseteq B$. Therefore $\Delta(a) \in B$ and hence $\Delta(I) \subseteq B$. Since B is the algebraic sum of all nilpotent ideals we have the result. ∎

In [3, Theorem 3.2] it is proved that if Φ is a continuous automorphism and Δ is a continuous Φ-derivation on a Banach algebra A and if J is an ideal of A, such that both Φ, $[\Delta, \Phi]$ leave J invariant, then $\Delta(J)/J$ is a quasinilpotent ideal of A/J. So, if J is a primitive ideal, then $\Delta(J)/J \subseteq Rad(A)/J = \{0\}$ and hence $\Delta(J) \subseteq J$. We use this fact in the proof of the next theorem.

Theorem 2.2. *Suppose that $\Phi, [\Delta, \Phi]$ leave each nilpotent and each primitive ideal invariant. If $S(\Delta)$ is nilpotent then $\Delta(P) \subseteq P$ for each primitive ideal P of A.*

Proof. $S(\Delta)$ is a nilpotent ideal, hence $S(\Delta) \subseteq B$. Let π be the canonical quotient map from A onto $A/\overline{B}$ then $\pi \circ \Delta$ is continuous. Therefore $\pi(\Delta(\overline{B})) = \{0\}$ and it follows that $\Delta(\overline{B}) \subseteq \overline{B}$. On the other hand, Φ leaves B invariant and Φ is continuous, thus Φ leaves $\overline{B}$ invariant and so it drops to a continuous automorphism $\Phi_0 : A/\overline{B} \to A/\overline{B}$. Consider $\Delta_0 : A/\overline{B} \to A/\overline{B}; \quad a + \overline{B} \mapsto \Delta(a) + \overline{B}$, which is a continuous Φ_0-derivation on $A/\overline{B}$ and by the argument just before this theorem $\Delta_0(P/\overline{B}) \subseteq P/\overline{B}$ for each primitive ideal P of A. Since $\overline{B} \subseteq P$ for every primitive ideal P, we have $\Delta(P) \subseteq P$. ■

Corollary 2.1. *If Φ is an inner automorphism on a Banach algebra A and if Δ is a Φ-derivation with a nilpotent separating ideal, then Δ leaves each primitive ideal invariant.*

Proof. Clearly for an inner automorphism Φ, $[\Delta, \Phi]$ leaves each ideal invariant. Now the result follows from Theorem 2.2. ■

3. Nilpotency of the Separating Ideal

Considering (Q1) we obtain some sufficient conditions for the separating ideal of a Φ-derivation on a Banach algebra to be nilpotent or quasinilpotent. Theorem 3.1 (ii) is a generalization of [5, Lemma 4.2] and Corollary 3.2 is [3, Corollary 4.3] which is proved in a different way. Theorem 3.3 and the other results of this section are generalizations of the results in [7]. Throughout this section by (A1) we mean the following assumption:

(A1). *The automorphism Φ is inner or Φ is continuous (as before), and $[\Delta, \Phi] = 0$.*

Under this assumption $S(\Delta)$ is invariant under $[\Delta, \Phi]$ and each $\Phi^j \quad (j \in Z)$.

Theorem 3.1. *Let A be a Banach algebra , and let Δ be a Φ-derivation on A. Set $J := S(\Delta) \cap R$. Then the following assertions hold.*

(i) *Let $Q(A)$ be the set of all quasinilpotent elements of A. If $\Delta(J) \subseteq Q(A)$, then $S(\Delta) \subseteq R$.*

(ii) *Assuming A(1) holds. If J is a nil ideal, then $S(\Delta)$ is a nilpotent ideal of A.*

Proof.

(i) Let $\Delta(J) \subseteq Q(A)$, but $S(\Delta) \nsubseteq R$. Since $S(\Delta)$ is a separating ideal, $S(\Delta)/J$ is finite dimensional by [6, Lemma 5.2.25]. Therefore $S(\Delta)$ has a strong Wederburn decomposition, that is there exists a finite dimensional subalgebra U of $S(\Delta)$ such that $S(\Delta) = U \oplus J$ and $S(\Delta)$ contains a nonzero idempotent, say e by [6, Theorem 2.8.6] . Let $\{a_n\}$ be a sequence in A, with $a_n \to 0$ and $\Delta(a_n) \to e$. Then $\{ea_n\} \subseteq S(\Delta)$ and there exist $\{u_n\} \subseteq U$ and $\{r_n\} \subseteq J$, such that $u_n \to 0, r_n \to 0$, and $ea_n = u_n + r_n$. We have $\Delta(ea_n) \to e$. Since U is finite dimensional $\Delta(u_n) \to 0$. Therefore $\Delta(r_n) \to e$, and so $e \in \overline{\Delta(J)} \subseteq \overline{Q(A)}$. Thus by [6, Corollary 2.4.8], the spectrum of e is a connected set containing the origin. It follows that the spectrum of e is nothing but the set $\{0\}$ and this contradicts the fact that e is non-zero. Thus $S(\Delta) \subseteq R$.

(ii) If J is nilpotent, then $J \subseteq B$. Suppose on the contrary that $S(\Delta)$ is not nilpotent, then $S(\Delta) \neq J$. Using the same notation as in the proof of (i), it follows that $e \in \overline{\Delta(J)} \subseteq \overline{\Delta(B)}$. Hence by (A1) and Theorem 2.1 $e \in \overline{B} \subseteq R$ which is a contradiction. ■

Corollary 3.2. *Each Φ-derivation Δ on a semisimple Banach algebra is continuous.*

Proof. As before let R denote the Jacobson radical. We have $S(\Delta) \cap R \subseteq R = \{0\}$ and by Theorem 3.1(i), $S(\Delta) \subseteq R$. Thus $S(\Delta) = \{0\}$, and Δ is continuous. ■

Theorem 3.3. *Let Δ be a Φ-derivation on a Banach algebra A such that $[\Delta, \Phi]$ and Φ are continuous. Let I be a closed ideal of A with $\Phi^{-1}(I) \subseteq I$. Then $S(\Delta) \cap I$ is nilpotent if and only if $\Delta^2\big|_{\overline{\bigcap_{n=1}^{\infty}(S(\Delta)\cap I)^n}}$ is continuous.*

Proof. We have $\Phi^{-1}(S(\Delta \cap I)) \subseteq S(\Delta) \cap I$. Suppose that Δ^2 is continuous on $\overline{\bigcap_{n=1}^{\infty}(S(\Delta) \cap I)^n}$. Consider $a \in S(\Delta) \cap I$, thus $\Phi^{-1}(a^n) = (\Phi^{-1}(a))^n \in \overline{(S(\Delta) \cap I)^n}$. Since $S(\Delta)$ is a separating ideal, there exists $N \in \mathbb{N}$ such that $\overline{S(\Delta)\Phi^{-1}(a^n)} = \overline{S(\Delta)\Phi^{-1}(a^N)}$ $(n \geq N)$. Hence by Mittag-Leffler theorem and the fact that $S(\Delta)\Phi^{-1}(a^n) \subseteq (S(\Delta) \cap I)^n$, we have

$$\overline{S(\Delta)\Phi^{-1}(a^N)} = \bigcap_{n=1}^{\infty} \overline{S(\Delta)\Phi^{-1}(a^n)} = \overline{\bigcap_{n=1}^{\infty} S(\Delta) \cap \Phi^{-1}(a^n)} \subseteq \overline{\bigcap_{n=1}^{\infty} (S(\Delta) \cap I)^n}.$$

Now, let $\{x_n\} \subseteq A$, $x_n \to 0$ and $\Delta(x_n) \to a^{N+1}$. Take $y_n = x_n\Phi^{-1}(a^{N+1})$, then $y_n \in S(\Delta)\Phi^{-1}(a^N) \subseteq \overline{\bigcap_{n=1}^{\infty}(S(\Delta) \cap I)^n}$, $y_n \to 0$, and $\Delta(y_n) = \Delta(x_n)a^{N+1} + x_n\Delta(\Phi^{-1}(a^{N+1})) \to a^{2(N+1)}$. Also by the hypothesis, $\Delta^2(y_n) \to 0$ and $\Delta^2(y_n\Phi^{-1}(y_n)) \to 0$. On the other hand, by the continuity of $[\Delta, \Phi]$

$$\Delta^2((y_n)\Phi^{-1}(y_n)) = (y_n)\Delta^2(\Phi^{-1}(y_n)) + \Delta(y_n)^2 + \Delta(y_n)\Phi(\Delta(\Phi^{-1}(y_n))) + \Delta^2(y_n)\Phi(y_n)$$

$\to 2a^{4(N+1)}$ as n tends to ∞. Therefore $a^{4(N+1)} = 0$, that is $S(\Delta) \bigcap I$ is a nil and hence a nilpotent ideal by closedness. The converse is trivial. ■

Note that the assumptions of Theorem 3.3 hold whenever Φ is inner.

Corollary 3.4. *Let Δ be a Φ-derivation on a Banach algebra A and let Φ satisfy (A1), then $S(\Delta)$ is a nilpotent ideal if and only if $\Delta^2\Big|_{\overline{\bigcap_{n=1}^{\infty}(S(\Delta)\cap R)^n}}$ is continuous.*

Proof. Since $\Phi^{-1}(R) \subseteq R$, then $S(\Delta) \cap R$ is nilpotent, by Theorem 3.3. Now Theorem 3.1 implies that $S(\Delta)$ is nilpotent. The converse is trivial. ■

Corollary 3.5. *Let Δ be a Φ-derivation on a Banach algebra A and let Φ satisfy (A1). If dim $(\bigcap_{n=1}^{\infty}(S(\Delta) \cap R)^n) < \infty$, then $S(\Delta)$ is nilpotent, and hence Δ leaves each primitive ideal of A invariant.*

Proof. This is immediate by Corollary 3.4 and Theorem 2.2. ■

Remark 3.6. Using the above results, the same notations and slightly different arguments as in [7], we observe that theorems 2.5, 2.6, 2.7 in [7] are also valid in the case of Φ-derivations whenever Φ satisfies assumption (A1). In particular, [7, Theorem 2.7] together with Corollary 2.1 above, show that "*if Φ is inner and the set $M(\Delta) = \{x \in S(\Delta) \cap R : \Delta(x) \in R\}$ is a nil set, then Δ leaves each primitive ideal invariant*".

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