# NON-LINEAR BENDING ANALYSIS OF LAMINATED COMPOSITE PLATES USING GENERALIZED DIFFERENTIAL QUADRATURE METHOD 

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## SUMMARY

Linear and non-linear bending analysis of laminated plates with different boundary conditions is presented using generalized differential quadrature (GDQ) method. Governing equations are based on the first-order shear deformation theory (FSDT) and von Kármán-type of geometric non-linearity. GDQ technique and Newton-Raphson method are employed to solve the system of non-linear equations.
Keywords: Non-linear analysis, Laminated plates, Generalized differential quadrature method, First-order shear deformation theory

## 1. INTRODUCTION

Composite laminated structures are being widely used in aerospace, automotive, marine and other technical applications. In reality, many plate structures are subjected to high load levels that may undergo large deflections. The effect of this large deflection is to stretch the middle plane of the plate inducing membrane stresses. By this membrane action, the load carrying capacity of the plate is increased to a large extent. For plates of this kind, the governing differential equations become non-linear. The non-linearity of the governing equations may be due to either material non-linearity or geometric nonlinearity. In this paper only geometric non-linearity will be considered. This nonlinearity is due to the fact that the strain displacement relations are non-linear.
Many approaches are used in the non-linear analysis of plates. Nath and Prithviraju [1] studied the non-linear statics and dynamics of laminated square plates. Their methodology of solution was based on the Chebyshev series technique. A comprehensive summary of the solutions for the geometrically non-linear analysis of isotropic and composite laminated plates is also given by Chia [2]. Semi-analytical approximations for the non-linear problem of laminated plates undergoing large deformation were also obtained using the Rayleigh-Ritz method [3] or the Galerkin method [4]. Yang and Shen [5] studied non-linear bending of shear deformable

[^0]functionally graded plates subjected to thermo-mechanical loads. Their formulations were based on Reddy's higher-order shear deformation plate theory with including thermal effects. In this study a semi-numerical approach, which makes use of multiparameter perturbation technique, one-dimensional differential quadrature approximation and Galerkin method, was employed to calculate the non-linear bending of the plates. Yang and Shen [6] also investigated the large deflection and postbuckling of functionally graded rectangular plates under transverse and in-plane loads.
The efforts of many authors are not only directed to accuracy and wide applicability of their formulations, but also to computational efficiency. Usually, accurate numerical solution of an engineering problem can be obtained by low-order finite difference and finite element methods using a large number of grid points. As a consequence, a lot of computational effort is needed. In seeking a more efficient method to get an accurate numerical solution by using just a few grid points, Bellman et al. [7,8] proposed a global method of differential quadrature (DQ) in 1972. The DQ approximates a derivative with respect to a coordinate direction at a grid point by a weighted linear sum of all the functional values in that direction. It was demonstrated that the DQ method was able to rapidly compute accurate solutions for partial differential equations by using only a few grid points in the respective solution domains [7,8]. Obviously, the first step for application of this method is to determine the weighting coefficients for any order partial derivatives. For the weighting coefficients of the first order derivatives, two techniques were suggested by Bellman et al. [7,8] which both have some restrictions. In order to overcome these difficulties, generalized differential quadrature was developed by Shu and Richards [9] and has been applied to solve some problems in fluid dynamics.
In this paper, generalized differential quadrature method is employed to obtain solutions for large deflection of moderately thick laminated plates. Accuracy and convergence of the method are examined with various examples.

## 2. GOVERNING EQUATIONS

Consider a rectangular plate of sides $a$ and $b$ and thickness $h$, shown in Figure 1. According to the first-order shear deformation theory, the displacement field at a point in the plate is expressed as [10]:
$U(x, y, z)=u(x, y)+z \phi_{x}(x, y)$
$V(x, y, z)=v(x, y)+z \phi_{y}(x, y)$
$W(x, y, z)=w(x, y)$
where $u, v$ and $w$ denote the displacements in $x, y$ and $z$ directions and $\phi_{x}$ and $\phi_{y}$ are rotation functions. The strain-displacement relations due to von Kármán-type of geometric non-linearity can be expressed as follows [10]:
$\varepsilon_{x}^{0}=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^{2}, \quad \varepsilon_{y}^{0}=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^{2}, \quad \varepsilon_{y z}^{0}=\frac{\partial w}{\partial y}+\phi_{y}$
$\varepsilon_{x z}^{0}=\frac{\partial w}{\partial x}+\phi_{x}, \quad \varepsilon_{x y}^{0}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)$
Underlined terms in Eqs. (2) are non-linear terms which are omitted for linear analysis. Also the curvatures are,


Figure 1: Laminate geometry and coordinate system
$\kappa_{x}^{0}=\frac{\partial \phi_{x}}{\partial x}, \quad \kappa_{y}^{0}=\frac{\partial \phi_{y}}{\partial y}, \quad \kappa_{x y}^{0}=\frac{\partial \phi_{y}}{\partial x}+\frac{\partial \phi_{x}}{\partial y}$
By using Eqs. (2) and (3) the constitutive equations are obtained as follows,

$$
\begin{align*}
& {\left[\begin{array}{l}
N_{x} \\
N_{y} \\
N_{x y}
\end{array}\right]=\left[\begin{array}{lll}
A_{11} & A_{12} & A_{16} \\
A_{21} & A_{22} & A_{26} \\
A_{61} & A_{62} & A_{66}
\end{array}\right]\left[\begin{array}{l}
\varepsilon_{x}^{0} \\
\varepsilon_{y}^{0} \\
\varepsilon_{x y}^{0}
\end{array}\right]+\left[\begin{array}{lll}
B_{11} & B_{12} & B_{16} \\
B_{21} & B_{22} & B_{26} \\
B_{61} & B_{62} & B_{66}
\end{array}\right]\left[\begin{array}{l}
\kappa_{x}^{0} \\
\kappa_{y}^{0} \\
\kappa_{x y}^{0}
\end{array}\right]}  \tag{4a}\\
& {\left[\begin{array}{l}
M_{x} \\
M_{y} \\
M_{x y}
\end{array}\right]=\left[\begin{array}{lll}
B_{11} & B_{12} & B_{16} \\
B_{21} & B_{22} & B_{26} \\
B_{61} & B_{62} & B_{66}
\end{array}\right]\left[\begin{array}{l}
\varepsilon_{x}^{0} \\
\varepsilon_{y}^{0} \\
\varepsilon_{x y}^{0}
\end{array}\right]+\left[\begin{array}{lll}
D_{11} & D_{12} & D_{16} \\
D_{21} & D_{22} & D_{26} \\
D_{61} & D_{62} & D_{66}
\end{array}\right]\left[\begin{array}{l}
\kappa_{x}^{0} \\
\kappa_{y}^{0} \\
\kappa_{x y}^{0}
\end{array}\right]}  \tag{4b}\\
& \left\{\begin{array}{l}
Q_{x} \\
Q_{y}
\end{array}\right\}=K_{s}\left[\begin{array}{ll}
A_{55} & A_{54} \\
A_{45} & A_{44}
\end{array}\right]\left\{\begin{array}{l}
\varepsilon_{x z}^{0} \\
\varepsilon_{y z}^{0}
\end{array}\right\} \tag{4c}
\end{align*}
$$

where $K_{s}$ is the shear correction factor and in all presented results $K_{s}=5 / 6$. Also

$$
\begin{equation*}
\left(A_{i j}, B_{i j}, D_{i j}\right)=\int_{-h / 2}^{h / 2} Q_{i j}\left(1, z, z^{2}\right) d z \quad i, j=1,2,6 \tag{5}
\end{equation*}
$$

Governing equations of the plate can be modified for functionally graded materials (FGMs). For FGMs we have [10]:
$Q_{11}=Q_{22}=\frac{E(z)}{1-v^{2}}, Q_{12}=Q_{21}=\frac{v E(z)}{1-v^{2}}, Q_{44}=Q_{55}=Q_{66}=\frac{E(z)}{2(1+v)}$,
$Q_{16}=Q_{61}=Q_{26}=Q_{62}=Q_{45}=Q_{54}=0$
$E(z)=\left(E_{c}-E_{m}\right)\left(\frac{z}{h}+\frac{1}{2}\right)^{n}+E_{m}$
In which $c$ and $m$ denote ceramic and metal, respectively.
Substituting Eqs. (1), (2) and (3) into constitutive equations in (4) leads to eight equations in terms of $u, v, w, \phi_{x}, \phi_{y}, N_{x}, N_{y}, N_{x y}, M_{x}, M_{y}, M_{x y}, Q_{x}$ and $Q_{y}$.

### 2.1. Equilibrium Equations

Equations of equilibrium can be derived using variational principle which is not explained in details here (see [10]). Five equilibrium equations are as follows,

$$
\begin{align*}
& \frac{\partial N_{x}}{\partial x}+\frac{\partial N_{x y}}{\partial y}=0, \quad \frac{\partial N_{x y}}{\partial x}+\frac{\partial N_{y}}{\partial y}=0 \\
& \frac{\partial Q_{x}}{\partial x}+\frac{\partial Q_{y}}{\partial y}+N_{x}\left(\frac{\partial^{2} w}{\partial x^{2}}\right)+N_{y}\left(\frac{\partial^{2} w}{\partial y^{2}}\right)+2 N_{x y}\left(\frac{\partial^{2} w}{\partial x \partial y}\right)+Q=0  \tag{7}\\
& \frac{\partial M_{x}}{\partial x}+\frac{\partial M_{x y}}{\partial y}-Q_{x}=0, \quad \frac{\partial M_{x y}}{\partial x}+\frac{\partial M_{y}}{\partial y}-Q_{y}=0
\end{align*}
$$

where $Q$ is the lateral distributed load. Five equilibrium equations in (7) together with eight constitutive equations in (4) are complete set of governing equations for the nonlinear bending of laminated plates which consists of thirteen partial differential equations.

### 2.1. Boundary Conditions

Different boundary conditions are considered at each edge of the plate as:

1. Clamped (C):
$u=v=w=\phi_{x}=\phi_{y}=0 \quad$ (at all edges)
2. Simply supported (S):

$$
\begin{array}{ll}
u=v=w=\phi_{y}=M_{x}=0 & \text { at }(x=\text { constant })  \tag{9a}\\
u=v=w=\phi_{x}=M_{y}=0 & \text { at }(y=\text { constant })
\end{array}
$$

3. Free (F):
$\begin{array}{ll}N_{x}=N_{x y}=Q_{x}=M_{x}=M_{x y}=0 & \text { at }(x=\text { constant }) \\ N_{y}=N_{x y}=Q_{y}=M_{y}=M_{x y}=0 & \text { at }(y=\text { constant })\end{array}$
In the following section, numerical solution procedure for the governing equations (4) and (7) subjected to a combination of boundary conditions in Eqs. (8)-(10) will be discussed in more details.

## 3. APPLICATION OF GDQ METHOD

The GDQ method is used to solve the non-linear differential equations of the plate. The plate is divided into $n_{x} \times n_{y}$ grid points where $n_{x}$ and $n_{y}$ represent the number of nodes in the $x$ and $y$ directions, respectively. Although the simplest procedure for discretization of the domain is to select equally spaced points, it is shown by Shu and Chew [11] that one of the best options for obtaining grid points is zeros of the well-known Chebyshev polynomials:
$x_{i}=\frac{a}{2}\left[1-\cos \left(\frac{i-1}{n-1} \pi\right)\right], \quad i=1,2, \ldots, n_{x}$
$y_{j}=\frac{b}{2}\left[1-\cos \left(\frac{j-1}{m-1} \pi\right)\right], \quad j=1,2, \ldots, n_{y}$
According to the GDQ method, the governing equations (4) and (7) can be re-written in discretized form. For example, equilibrium equations (7) at a sample grid point $\left(x_{i}, y_{j}\right)$ can be written as:

$$
\begin{align*}
& \sum_{k=1}^{n_{x}} A_{i k} N_{x}\left(x_{k}, y_{j}\right)+\sum_{k=1}^{n_{y}} \bar{A}_{j k} N_{x y}\left(x_{i}, y_{k}\right)=0 \\
& \sum_{k=1}^{n_{x}} A_{i k} N_{x y}\left(x_{k}, y_{j}\right)+\sum_{k=1}^{n_{y}} \bar{A}_{j k} N_{y}\left(x_{i}, y_{k}\right)=0 \\
& \sum_{k=1}^{n_{x}} A_{i k} Q_{x}\left(x_{k}, y_{j}\right)+\sum_{k=1}^{n_{y}} \bar{A}_{j k} Q_{y}\left(x_{i}, y_{k}\right)+N_{x}\left(x_{i}, y_{j}\right) \sum_{k=1}^{n_{x}} B_{i k} w\left(x_{k}, y_{j}\right) \\
& \quad+N_{y}\left(x_{i}, y_{j}\right) \sum_{k=1}^{n_{y}} \bar{B}_{j k} w\left(x_{i}, y_{k}\right)+2 N_{x y}\left(x_{i}, y_{j}\right) \sum_{k=1}^{n_{x}} \sum_{l=1}^{n_{y}} \bar{A}_{j l} A_{i k} w\left(x_{k}, y_{l}\right)+Q=0  \tag{12}\\
& \sum_{k=1}^{n_{x}} A_{i k} M_{x}\left(x_{k}, y_{j}\right)+\sum_{k=1}^{n_{y}} \bar{A}_{j k} M_{x y}\left(x_{i}, y_{k}\right)-Q_{x}\left(x_{i}, y_{j}\right)=0 \\
& \sum_{k=1}^{n_{x}} A_{i k} M_{x y}\left(x_{k}, y_{j}\right)+\sum_{k=1}^{n_{y}} \bar{A}_{j k} M_{y}\left(x_{i}, y_{k}\right)-Q_{y}\left(x_{i}, y_{j}\right)=0
\end{align*}
$$

where $\left(x_{i}, y_{j}\right)$ is a grid point inside the plate with $i=2, \ldots, n_{x}-1$ and $j=2, \ldots, n_{y}-1 . A_{i j}$ and $\bar{A}_{i j}$ are weighting coefficients for first-order partial derivatives and $B_{i j}$ and $\bar{B}_{i j}$ for second-order partial derivatives [11].
Following the procedure explained above leads to a system of $13\left(n_{x} \times n_{y}\right)$ non-linear algebraic equations with the same number of unknowns.
Note that applying boundary conditions to the obtained algebraic equations, five of the thirteen unknown parameters at each boundary node will vanish. At last an incrementaliterative method should be used to solve the resulting non-linear system of equations. In the present analysis, the solution algorithms are based on the Newton-Raphson method.

## 4. RESULTS AND DISCUSSION

The presented GDQ method together with Newton-Raphson iterative scheme is used to obtain solution to the governing equations of linear/non-linear bending of laminated plates with different combination of boundary conditions: simply supported (S), clamped (C) and free edges (F). The edges of the plate are numbered from 1 to 4 as shown in Figure 1. For example the symbol SCCS, identifies a plate with edge 1 simply supported, edges 2 and 3 clamped and edge 4 simply supported. In all examples material properties of the plate are:
Graphite-epoxy: $\frac{E_{11}}{E_{22}}=25, \quad \frac{G_{12}}{E_{22}}=\frac{G_{13}}{E_{22}}=0.5, \quad \frac{G_{23}}{E_{22}}=0.2, \quad v_{12}=0.25$
In this study, two types of loading including uniform and sinusoidal distributed loads are considered as:
Uniform load:

$$
\begin{equation*}
Q_{(x, y)}=Q_{0} \tag{14}
\end{equation*}
$$

Sinusoidal load: $\quad Q_{(x, y)}=Q_{0} \sin (\pi x / a) \sin (\pi y / b)$
All results presented in this section are compared with results of other numerical and analytical studies available in the literature. In certain cases where results were not found in the open literature all predictions are compared with results of the commercial finite element code ABAQUS.

### 4.1. Linear Analysis

First examples regards to linear analysis of [0/90] and [0/90] $]_{10}$ laminated square plates of side $a$ with two opposite straight edges simply-supported subjected to sinusoidally distributed loading. The other two edges of the plates could be any combination of clamped, free or simply supported. Side to thickness ratios of the plates are $a / h=5$ and 10.

Table 1 compares the normalized central deflection of [0/90] laminated square plate with analytical and FEM solutions [10] based on the first- and third-order shear deformation plate theories together with classical laminate plate theory (CLPT). Table 2 also presents the normalized central deflection of $[0 / 90]_{10}$ laminated square plate. All numerical results of Tables 1 and 2 for central deflection are normalized using $w^{*}=100 w \times E_{22} h^{3} / q_{0} a^{4}$. Results presented in both Tables reveal that assuming large and small values for geometry parameters do not affect accuracy of the results of GDQ method as they very well match with analytical solutions.

Table 1: Normalized central deflection of [0/90] graphite-epoxy square plate under sinusoidally loading (linear analysis).

| BC | $a / h$ | Present GDQ |  |  |  |  | $\begin{aligned} & \text { FSDT } \\ & \text { Exact } \\ & \text { [10] } \end{aligned}$ | $\begin{aligned} & \text { FSDT } \\ & \text { FEM } \\ & \text { [10] } \end{aligned}$ | $\begin{aligned} & \text { TSDT } \\ & \text { [10] } \end{aligned}$ | $\begin{aligned} & \text { CLPT } \\ & \text { [10] } \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $5 \times 5$ | $7 \times 7$ | $9 \times 9$ | $11 \times 11$ | $13 \times 13$ |  |  |  |  |
| FF | 5 | 2.727 | 2.770 | 2.777 | 2.777 | 2.777 | 2.777 | 2.776 | 2.624 | 1.777 |
|  | 10 | 1.990 | 2.019 | 2.025 | 2.028 | 2.028 | 2.028 | 2.027 | 1.992 | 1.777 |
| FS | 5 | 2.298 | 2.334 | 2.335 | 2.335 | 2.335 | 2.335 | 2.334 | 2.211 | 1.471 |
|  | 10 | 1.654 | 1.686 | 1.687 | 1.687 | 1.687 | 1.687 | 1.687 | 1.658 | 1.471 |
| FC | 5 | 1.872 | 1.901 | 1.897 | 1.897 | 1.897 | 1.897 | 1.897 | 1.733 | 0.980 |
|  | 10 | 1.151 | 1.227 | 1.223 | 1.223 | 1.223 | 1.223 | 1.223 | 1.184 | 0.980 |
| SS | 5 | 1.730 | 1.758 | 1.758 | 1.758 | 1.758 | 1.758 | 1.759 | 1.667 | 1.064 |
|  | 10 | 1.206 | 1.236 | 1.237 | 1.237 | 1.237 | 1.237 | 1.238 | 1.216 | 1.064 |
| SC | 5 | 1.448 | 1.479 | 1.477 | 1.477 | 1.477 | 1.477 | 1.478 | 1.333 | 0.664 |
|  | 10 | 0.805 | 0.888 | 0.883 | 0.883 | 0.883 | 0.883 | 0.883 | 0.848 | 0.664 |
| CC | 5 | 1.225 | 1.261 | 1.257 | 1.257 | 1.257 | 1.257 | 1.257 | 1.088 | 0.429 |
|  | 10 | 0.558 | 0.664 | 0.656 | 0.656 | 0.656 | 0.656 | 0.657 | 0.617 | 0.429 |

### 4.1. Non-Linear Analysis

The convergence behavior and accuracy of the method in non-linear static analysis are also checked for laminated plates. It is assumed that the plates are subjected to uniform distributed load in all examples.
The central deflection of a [0/90/90/0] laminated square plate is validated with those predicted by Zhang and Kim [12] using the finite element method, shown in Table 3. Zhang and Kim [12] also employed FSDT for their non-linear analysis. In this table the
boundary conditions of the plate is considered to be CCCC and side to thickness ratio $a / h=40$.

Table 2: Normalized central deflection of $[0 / 90]_{10}$ graphite-epoxy square plate under sinusoidally loading (linear analysis).

| BC $a / h$ |  | Present GDQ |  |  |  |  | FSDT <br> Exact <br> [10] | $\begin{aligned} & \text { FSDT } \\ & \text { FEM } \\ & {[10]} \end{aligned}$ | $\begin{aligned} & \text { TSDT } \\ & \text { [10] } \end{aligned}$ | $\begin{aligned} & \text { CLPT } \\ & \text { [10] } \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $5 \times 5$ | $7 \times 7$ | $9 \times 9$ | $11 \times 11$ | $13 \times 13$ |  |  |  |  |
| FF | 5 | 1.643 | 1.663 | 1.663 | 1.663 | 1.663 | 1.663 | 1.662 | 1.651 | 0.665 |
|  | 10 | 0.900 | 0.914 | 0.915 | 0.915 | 0.915 | 0.915 | 0.914 | 0.916 | 0.665 |
| FS | 5 | 1.439 | 1.459 | 1.460 | 1.460 | 1.460 | 1.460 | 1.460 | 1.450 | 0.579 |
|  | 10 | 0.787 | 0.800 | 0.800 | 0.800 | 0.800 | 0.800 | 0.800 | 0.801 | 0.579 |
| FC | 5 | 0.251 | 1.258 | 1.258 | 1.258 | 1.258 | 1.258 | 1.258 | 1.214 | 0.380 |
|  | 10 | 0.609 | 0.612 | 0.612 | 0.612 | 0.612 | 0.612 | 0.612 | 0.607 | 0.380 |
| SS | 5 | 1.121 | 1.136 | 1.137 | 1.137 | 1.137 | 1.137 | 1.137 | 1.129 | 0.442 |
|  | 10 | 0.605 | 0.615 | 0.615 | 0.615 | 0.615 | 0.615 | 0.616 | 0.616 | 0.442 |
| SC | 5 | 1.033 | 1.045 | 1.045 | 1.045 | 1.045 | 1.045 | 1.045 | 1.001 | 0.266 |
|  | 10 | 0.473 | 0.480 | 0.780 | 0.480 | 0.480 | 0.480 | 0.480 | 0.473 | 0.266 |
| CC | 5 | 0.937 | 0.945 | 0.945 | 0.945 | 0.945 | 0.945 | 0.945 | 0.879 | 0.167 |
|  | 10 | 0.379 | 0.385 | 0.385 | 0.385 | 0.385 | 0.385 | 0.386 | 0.375 | 0.167 |

Table 4 also represents normalized maximum deflection due to different values of load parameter of CCSS laminated square plate with side to thickness ratio of $a / h=10$. Included in the table is also results of finite element code ABAQUS.
Non-dimensional deflection of [0/90] laminated square plate on its center line is studied in Figures 2(a) and 2(b). In these figures the boundary conditions of the plate is considered to be SSSC and SCSF, respectively.
Figure 3(a) shows the effect of boundary conditions on the deflection of the [0/90] laminated square plate. Deflection of the FFCC laminated square plate is also presented for two different laminate lay-ups in Figure 3(b). In Figures 2 and 3 side to thickness ratio of the plate is $a / h=10$ and $Q_{0} a^{4} / E_{22} h^{4}=100$.
In the previous section, it is noticed that governing equations can be modified for functionally graded plates in order to examine the GDQ method for non-linear bending of FG plates. So the last example regards to non-linear bending of Aluminum-Alumina FG plates. Normalized central deflection of this plate versus load is shown in Figure 4(a). Included in these figure is also analytical results of GhannadPour and Alinia [13]. It is seen that the GDQ results are in good agreement with those obtained by analytical solution. Figure 4(b) also presents variation of normalized deflection of CCSS square FG plate at $y / b=0.5$. In Figure 4(a) $a / h=10$ and in Figure 4(b) $Q_{0} a^{4} / E_{22} h^{4}=200$. It is clear that the results of present GDQ method are in good agreement with ABAQUS finite element results and good accuracy of the GDQ method is a noticeable point of all examples.

Table 3: Normalized deflection ( $w / h$ ) of CCCC graphite-epoxy square plate (non-linear analysis).

| $\frac{Q_{0} a^{4}}{E_{22} 4^{4}}$ | Present work |  |  | Zhang and Kim [12] |
| :--- | :--- | :--- | :--- | :--- |
|  | $7 \times 7$ | $9 \times 9$ | $11 \times 11$ |  |
| 100 | 0.4725 | 0.4704 | 0.4703 | 0.4608 |
| 150 | 0.5882 | 0.5852 | 0.5851 | 0.5771 |
| 200 | 0.6846 | 0.6758 | 0.6757 | 0.6668 |
| 250 | 0.7601 | 0.7549 | 0.7547 | 0.7403 |

Table 4: Normalized deflection ( $w / h$ ) of CCSS graphite-epoxy square plate (non-linear analysis).

| $Q_{0} a^{4}$ | Present work |  |  | ABAQUS |
| :--- | :--- | :--- | :--- | :--- |
|  | $9 \times 9$ | $11 \times 11$ | $13 \times 13$ |  |
| 100 | 0.70129 | 0.74193 | 0.74194 | 0.75436 |
| 200 | 0.95650 | 1.11040 | 1.11041 | 1.13099 |
| 300 | 1.30495 | 1.35439 | 1.35439 | 1.38000 |
| 400 | 1.49655 | 1.54173 | 1.54174 | 1.56001 |
| 500 | 1.62840 | 1.68500 | 1.68501 | 1.70345 |




Figure 2: Normalized deflection of graphite-epoxy square plate with (a) SSSC and (b) SCSF boundary conditions (non-linear analysis).


Figure 3: Effect of (a) boundary conditions and (b) laminate lay-ups on the deflection of [0/90] graphite-epoxy square plate (non-linear analysis).


Figure 4: (a) Normalized central deflection of SSSS FG square plate versus load. (b)Variation of normalized deflection of CCSS FG square plate at $y / b=0.5$ (non-linear analysis).

## 5. CONCLUSIONS

The generalized differential quadrature method is used to obtain numerical solution for linear and non-linear bending of laminated square plates subjected to sinusoidally/uniformly distributed load and various boundary conditions. Comparisons of the results with those available in the literature and results of finite element code ABAQUS show good agreements. It is shown that the system equation which is used in this study provides a simple procedure to apply various boundary conditions. Results also revealed that the method is efficient and accurate and therefore, could be used for more complicated problems.

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