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A Characterization of Gorenstein Rings and Grothendieck's Conjecture*

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Abstract. Given a commutative Noetherian local ring $(R, \mathfrak{m})$, it is shown that R is Gorenstein if and only if there exists a system of parameters $x_1, \ldots, x_d$ of R which generates an irreducible ideal and

$$\sum_{j=1}^{d} x_j^t R :_R \mathfrak{m} \subseteq \Big(\bigcup_{s \in \mathbb{N}} \Big(\sum_{j=1}^{d} x_j^{t+s} R \Big) :_R x_1^s \cdots x_d^s \Big) + x_1^{t-1} \cdots x_d^{t-1} \Big(\sum_{j=1}^{d} x_j R :_R \mathfrak{m} \Big)$$

for all $t > 0$. Let n be an arbitrary non-negative integer. It is also shown that for an arbitrary ideal $\mathfrak{a}$ of a commutative Noetherian (not necessarily local) ring R and a finitely generated R-module M, $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^n(M))$ is finitely generated if and only if there exists an $\mathfrak{a}$-filter regular sequence $x_1, \ldots, x_n \in \mathfrak{a}$ such that

$$\sum_{j=1}^{n} x_j^t M :_M \mathfrak{a} \subseteq \Big(\bigcup_{s \in \mathbb{N}} \Big(\sum_{j=1}^{n} x_j^{t+s} M \Big) :_M x_1^s \cdots x_n^s \Big) + x_1^{t-1} \cdots x_n^{t-1} \Big(\sum_{j=1}^{n} x_j M :_M \mathfrak{a} \Big)$$

for all $t > 0$.

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1 Introduction

Let R be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$. It is well known that a Cohen–Macaulay ring R is Gorenstein if and only if some ideal generated by a system of parameters (called a parameter ideal) is irreducible. Also, it

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follows from a result of Northcott and Rees [13, Theorem 1] that if every parameter ideal is irreducible, then R is Cohen–Macaulay. Therefore, R is Gorenstein if and only if every parameter ideal is irreducible. Recently, in [11], Marley, Rogers and Sakurai proved that there exists an integer ℓ such that R is Gorenstein if and only if some parameter ideal contained in $\mathfrak{m}^{\ell}$ is irreducible. They also showed that the integer ℓ identified in their result may be taken to be the least integer $\delta = \delta(R)$ such that the canonical map

$$\mathrm{Ext}^d_R(R/\mathfrak{m}^{\delta}, R) \longrightarrow \varinjlim_{\alpha} \mathrm{Ext}^d_R(R/\mathfrak{m}^{\alpha}, R) \cong H^d_{\mathfrak{m}}(R)$$

is surjective after applying the functor $\mathrm{Hom}_R(R/\mathfrak{m}, -)$, where $d = \dim R$. Moreover, they proved that in the present situation, the map

$$\mathrm{Hom}_R\Big(R/\mathfrak{m}, \frac{R}{(x_1,\ldots,x_d)}\Big) \longrightarrow \mathrm{Hom}(R/\mathfrak{m}, H^d_{\mathfrak{m}}(R))$$

which is induced by the canonical map

$$\frac{R}{(x_1,\ldots,x_d)} \longrightarrow \varinjlim_{\alpha} \frac{R}{(x_1^{\alpha},\ldots,x_d^{\alpha})}$$

is surjective, where $x_1,\ldots,x_d$ is a system of parameters of R such that $(x_1,\ldots,x_d)$ is an irreducible ideal contained in $\mathfrak{m}^{\ell}$ (see also [4, Lemma 3.12]). In this paper, for a commutative Noetherian (not necessarily local) ring R, a finitely generated R-module M and an arbitrary sequence $x_1,\ldots,x_n$ of elements of R, we show that the map

$$\varphi_{\mathfrak{a},x} : \mathrm{Hom}_R\Big(R/\mathfrak{a}, \frac{M}{(x_1,\ldots,x_n)M}\Big) \longrightarrow \mathrm{Hom}_R\big(R/\mathfrak{a}, H^n_{(x_1,\ldots,x_n)}(M)\big)$$

which is induced by the canonical map

$$\frac{M}{(x_1,\ldots,x_n)M} \longrightarrow \varinjlim_{\alpha} \frac{M}{(x_1^{\alpha},\ldots,x_n^{\alpha})M} \cong H^n_{(x_1,\ldots,x_n)}(M)$$

is surjective if and only if

$$\sum_{j=1}^{n} x_j^t M :_M \mathfrak{a} \subseteq \Big(\bigcup_{s\in\mathbb{N}} \Big(\sum_{j=1}^{n} x_j^{t+s} M\Big) :_M x_1^s \cdots x_n^s\Big) + x_1^{t-1}\cdots x_n^{t-1}\Big(\sum_{j=1}^{n} x_j M :_M \mathfrak{a}\Big)$$

for all $t \in \mathbb{N}$ (we use $\mathbb{N}$ to denote the set of positive integers). In the light of the ideas of Marley, Rogers and Sakurai's proof of [11, Theorem 2.9], by using our above mentioned result, we can obtain a characterization of Gorenstein rings (see Theorem 2.2).

On the other hand, if $(R,\mathfrak{m})$ is a Noetherian local ring and M is a finitely generated R-module, it is well known that for all i and n, $\mathrm{Supp}_R(H^n_{\mathfrak{m}}(M)) \subseteq V(\mathfrak{m})$ and $\mathrm{Ext}^i_R(R/\mathfrak{m}, H^n_{\mathfrak{m}}(M))$ is finitely generated (see, e.g., Huneke and Koh [8]). In this regard, Grothendieck made the following:

Conjecture. [5] If $\mathfrak{a}$ is an ideal of R and M is a finitely generated R-module, then $\operatorname{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$ is finitely generated for all n.

Hartshorne [6] has produced a counterexample which shows that this conjecture is false even when R is regular (see also [7]). Hartshorne asked the following:

Question. If $\mathfrak{a}$ is an ideal of R and M is a finitely generated R-module, when are $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$ finitely generated for all n and j?

There are several papers devoted to obtain partial answer to Hartshorne's question. We refer the reader to Huneke and Koh [8], Delfino [2], Delfino and Marley [3], Yoshida [16] and the present author [9].

In Theorem 2.4 of this paper, by using a natural generalization of the concept of regular sequence, we show that for a fixed n, Grothendieck's conjecture is true if and only if there exists a certain sequence of elements in $\mathfrak{a}$. In fact, we present a new version of Grothendieck's conjecture in commutative algebra.

Throughout this paper, R will denote a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ an ideal of R. Also, M will denote a finitely generated R-module. Our terminology follows the textbook [1] on local cohomology. Whenever we can do without ambiguity, for a sequence $x = x_1, \ldots, x_n$ of elements of R and $u \in \mathbb{N}$, we will denote $x_1^u, \ldots, x_n^u$ by x^u.

2 Certain Sequence of Elements of R

Let $x = x_1, \ldots, x_n$ be a sequence of elements of R. It follows from [5, Theorem 2.8] that the n-th local cohomology module $H^n_{(x_1,\ldots,x_n)}(M)$ can be interpreted as a direct limit of Koszul homology modules, and in the present situation we have

$$H^n_{(x_1,\ldots,x_n)}(M) \cong \varinjlim_{\alpha\in\mathbb{N}} \frac{M}{(x_1^\alpha, \ldots, x_n^\alpha)M}$$

with the map

$$\frac{M}{(x_1^u, \ldots, x_n^u)M} \longrightarrow \frac{M}{(x_1^v, \ldots, x_n^v)M}$$

being induced by multiplication by $x_1^{v-u} \cdots x_n^{v-u}$ for all $u, v \in \mathbb{N}$ with $1 \leqslant u \leqslant v$. Therefore,

$$\operatorname{Hom}_R\big(R/\mathfrak{a},\, H^n_{(x_1,\ldots,x_n)}(M)\big) \cong \varinjlim_{\alpha\in\mathbb{N}} \operatorname{Hom}_R\Big(R/\mathfrak{a},\, \frac{M}{(x_1^\alpha, \ldots, x_n^\alpha)M}\Big).$$

For $\alpha \in \mathbb{N}$, we denote by $\varphi_{\mathfrak{a},x^\alpha}$ the map

$$\operatorname{Hom}_R\Big(R/\mathfrak{a},\, \frac{M}{(x_1^\alpha, \ldots, x_n^\alpha)M}\Big) \longrightarrow \operatorname{Hom}_R\big(R/\mathfrak{a},\, H^n_{(x_1,\ldots,x_n)}(M)\big)$$

which is induced by the canonical map

$$\frac{M}{(x_1^\alpha, \ldots, x_n^\alpha)M} \longrightarrow \varinjlim_{\alpha} \frac{M}{(x_1^\alpha, \ldots, x_n^\alpha)M}\,.$$

In the following lemma, we show that $\varphi_{\mathfrak{a},x}$ is surjective if and only if $x_1, \ldots, x_n$ satisfy certain conditions.

Lemma 2.1. *For a sequence $x = x_1, \ldots, x_n$ of elements of R, the map $\varphi_{\mathfrak{a},x}$ is surjective if and only if*

$$\sum_{j=1}^{n} x_j^t M :_M \mathfrak{a} \subseteq \Big(\bigcup_{s\in\mathbb{N}} \Big(\sum_{j=1}^{n} x_j^{t+s} M\Big) :_M x_1^s \cdots x_n^s\Big) + x_1^{t-1} \cdots x_n^{t-1} \Big(\sum_{j=1}^{n} x_j M :_M \mathfrak{a}\Big) \quad (1)$$

for all $t \in \mathbb{N}$.

Proof. First of all, let $x = x_1, \ldots, x_n$ be a sequence of elements of R. Since

$$\operatorname{Hom}_R\Big(R/\mathfrak{a}, \frac{M}{(x_1, \ldots, x_n)M}\Big) \cong \frac{(x_1, \ldots, x_n)M :_M \mathfrak{a}}{(x_1, \ldots, x_n)M},$$

we have the following commutative diagram:

$$\begin{array}{ccc}
\operatorname{Hom}_R\Big(R/\mathfrak{a}, \dfrac{M}{(x_1, \ldots, x_n)M}\Big) & \xrightarrow{\varphi_{\mathfrak{a},x}} & \operatorname{Hom}_R\big(R/\mathfrak{a}, H^n_{(x_1,\ldots,x_n)}(M)\big) \\
\downarrow\cong & & \downarrow\cong \\
\dfrac{(x_1, \ldots, x_n)M :_M \mathfrak{a}}{(x_1, \ldots, x_n)M} & \xrightarrow{\psi_{\mathfrak{a},x}} & \varinjlim_{\alpha\in\mathbb{N}} \dfrac{(x_1^\alpha, \ldots, x_n^\alpha)M :_M \mathfrak{a}}{(x_1^\alpha, \ldots, x_n^\alpha)M}
\end{array}$$

where the direct system $\Big\{\frac{(x_1^\alpha,\ldots,x_n^\alpha)M :_M \mathfrak{a}}{(x_1^\alpha,\ldots,x_n^\alpha)M}\Big\}_{\alpha\in\mathbb{N}}$ is given by the map induced by multiplication

$$\frac{(x_1^u, \ldots, x_n^u)M :_M \mathfrak{a}}{(x_1^u, \ldots, x_n^u)M} \xrightarrow{x_1^{v-u}\cdots x_n^{v-u}} \frac{(x_1^v, \ldots, x_n^v)M :_M \mathfrak{a}}{(x_1^v, \ldots, x_n^v)M}$$

for $u, v \in \mathbb{N}$ with $1 \leqslant u \leqslant v$, and $\psi_{\mathfrak{a},x}$ is the canonical map.

Now suppose that $\varphi_{\mathfrak{a},x}$ is surjective and $m \in \sum_{j=1}^n x_j^t M :_M \mathfrak{a}$, where $t \in \mathbb{N}$. In view of the above commutative diagram, $\psi_{\mathfrak{a},x}$ is also surjective. Hence, there exists $m' \in \sum_{j=1}^n x_j M :_M \mathfrak{a}$ such that

$$\psi_{\mathfrak{a},x}\big(m' + (x_1, \ldots, x_n)M\big) = \psi_{\mathfrak{a},x^t}\big(m + (x_1^t, \ldots, x_n^t)M\big).$$

This implies that the element

$$\psi_{\mathfrak{a},x^t}\big(m - x_1^{t-1} \cdots x_n^{t-1} m' + (x_1^t, \ldots, x_n^t)M\big)$$

is zero in $\varinjlim_{\alpha\in\mathbb{N}} \frac{(x_1^\alpha,\ldots,x_n^\alpha)M :_M \mathfrak{a}}{(x_1^\alpha,\ldots,x_n^\alpha)M}$. Thus, there exists $s \in \mathbb{N}$ such that

$$x_1^s \cdots x_n^s (m - x_1^{t-1} \cdots x_n^{t-1} m') \in \sum_{j=1}^{n} x_j^{t+s} M$$

and so

$$m \in \Big(\bigcup_{s\in\mathbb{N}}\Big(\sum_{j=1}^{n} x_j^{t+s}M\Big) :_M x_1^s\cdots x_n^s\Big) + x_1^{t-1}\cdots x_n^{t-1}\Big(\sum_{j=1}^{n} x_jM :_M \mathfrak{a}\Big).$$

Conversely, suppose that $x = x_1,\dots,x_n$ is a sequence of elements of R such that the inclusion (1) holds for all $t\in\mathbb{N}$. We must show that $\varphi_{\mathfrak{a},x}$ is surjective. By using the above commutative diagram, it is enough to show that $\psi_{\mathfrak{a},x}$ is surjective. To this end, let ξ be an arbitrary element of $\varinjlim_{\alpha\in\mathbb{N}} \frac{(x_1^\alpha,\dots,x_n^\alpha)M :_M \mathfrak{a}}{(x_1^\alpha,\dots,x_n^\alpha)M}$. Then there exists a positive integer t such that

$$\xi = \psi_{\mathfrak{a},x^t}\big(m + (x_1^t,\dots,x_n^t)M\big)$$

for some $m \in \sum_{j=1}^n x_j^tM :_M \mathfrak{a}$. By our assumption, $m = m_1 + x_1^{t-1}\cdots x_n^{t-1}m_2$ for some

$$m_1 \in \Big(\bigcup_{s\in\mathbb{N}}\Big(\sum_{j=1}^{n} x_j^{t+s}M\Big) :_M x_1^s\cdots x_n^s\Big)$$

and $m_2 \in \sum_{j=1}^n x_jM :_M \mathfrak{a}$. Therefore, $m_1 \in \sum_{j=1}^n x_j^{t+s}M :_M x_1^s\cdots x_n^s$ for some $s\in\mathbb{N}$. We will show that

$$\xi = \psi_{\mathfrak{a},x}\big(m_2 + (x_1,\dots,x_n)M\big).$$

Thus, it suffices to show that $\psi_{\mathfrak{a},x^t}\big(m - x_1^{t-1}\cdots x_n^{t-1}m_2 + (x_1^t,\dots,x_n^t)M\big) = 0$. This is clear because

$$\begin{aligned}
&\psi_{\mathfrak{a},x^t}\big(m - x_1^{t-1}\cdots x_n^{t-1}m_2 + (x_1^t,\dots,x_n^t)M\big)\\
=\;& \psi_{\mathfrak{a},x^t}\big(m_1 + (x_1^t,\dots,x_n^t)M\big)\\
=\;& \psi_{\mathfrak{a},x^{t+s}}\big(x_1^s\cdots x_n^s m_1 + (x_1^{t+s},\dots,x_n^{t+s})M\big) = 0,
\end{aligned}$$

which completes the proof. □

The following theorem is one of the main results in this paper. Its proof relies heavily on ideas of Marley, Rogers and Sakurai's proof of [11, Theorem 2.9].

Theorem 2.2. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension d. Then the following conditions are equivalent:*

(i) *R is Gorenstein.*

(ii) *There exists an integer ℓ such that some parameter ideal contained in $\mathfrak{m}^\ell$ is irreducible.*

(iii) *There exists a system of parameters $x_1,\dots,x_d$ of R such that $(x_1,\dots,x_d)$ is an irreducible parameter ideal and for all $t\in\mathbb{N}$,*

$$\sum_{j=1}^{d} x_j^tR :_R \mathfrak{m} \subseteq \Big(\bigcup_{s\in\mathbb{N}}\Big(\sum_{j=1}^{d} x_j^{t+s}R\Big) :_R x_1^s\cdots x_d^s\Big) + x_1^{t-1}\cdots x_d^{t-1}\Big(\sum_{j=1}^{d} x_jR :_R \mathfrak{m}\Big). \quad (2)$$

Proof. The equivalence (i)$\Leftrightarrow$(ii) is proved in [11, Theorem 2.9].

(ii)$\Rightarrow$(iii) Let $x = x_1, \ldots, x_d$ be a system of parameters in $\mathfrak{m}^\ell$ which generates an irreducible ideal. By [11, Proposition 2.8], the map $\varphi_{\mathfrak{m},x}$ is surjective. The result now follows from Lemma 2.1.

(iii)$\Rightarrow$(i) It suffices to show that if there exists a system of parameters $x = x_1, \ldots, x_d$ which generates an irreducible ideal and satisfies the inclusion (2) for all $t \in \mathbb{N}$, then R is Cohen–Macaulay (and hence Gorenstein). In view of Lemma 2.1, by our assumption, the map

$$\operatorname{Hom}_R\Big(R/\mathfrak{m}, \frac{R}{(x_1, \ldots, x_d)}\Big) \xrightarrow{\varphi_{\mathfrak{m},x}} \operatorname{Hom}_R\big(R/\mathfrak{m}, H^d_{(x_1,\ldots,x_d)}(R)\big) \cong \operatorname{Hom}_R(R/\mathfrak{m}, H^d_{\mathfrak{m}}(R))$$

is surjective. By employing a method of proof which is similar to that used in [11, Theorem 2.9], it is easy to see that $x_1, \ldots, x_d$ is a regular sequence and so R is Cohen–Macaulay. $\square$

Now we mention a generalization of the concept of regular sequences which is needed in the rest of the paper. We say that a sequence $x_1, \ldots, x_n$ of elements of $\mathfrak{a}$ is an $\mathfrak{a}$-filter regular sequence on M if

$$\operatorname{Supp}_R\left(\frac{(x_1, \ldots, x_{i-1})M :_M x_i}{(x_1, \ldots, x_{i-1})M}\right) \subseteq V(\mathfrak{a})$$

for all $i = 1, \ldots, n$, where $V(\mathfrak{a})$ denotes the set of prime ideals of R containing $\mathfrak{a}$. The concept of an $\mathfrak{a}$-filter regular sequence on M is a generalization of the concept of a filter regular sequence which has been studied in [14, 15] and has led to some interesting results. Both concepts coincide if $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal of a local ring with maximal ideal $\mathfrak{m}$. Note that $x_1, \ldots, x_n$ is a weak M-sequence if and only if it is an R-filter regular sequence on M. It is easy to see that the analogue of [15, Appendix 2(ii)] holds true whenever R is Noetherian, M is finitely generated and $\mathfrak{m}$ is replaced by $\mathfrak{a}$, so if $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M, then there is an element $y \in \mathfrak{a}$ such that $x_1, \ldots, x_n, y$ is an $\mathfrak{a}$-filter regular sequence on M. Thus, for any positive integer n, there exists an $\mathfrak{a}$-filter regular sequence on M of length n.

The following proposition comes from [10, Proposition 1.2] and [12, Lemma 3.4].

Proposition 2.3. *Let $n > 0$, and $x_1, \ldots, x_n$ be an $\mathfrak{a}$-filter regular sequence on M. Then there are the following isomorphisms:*

$$H^i_{\mathfrak{a}}(M) \cong \begin{cases} H^i_{(x_1,\ldots,x_n)}(M) & \text{for } 0 \leqslant i < n, \\ H^{i-n}_{\mathfrak{a}}(H^n_{(x_1,\ldots,x_n)}(M)) & \text{for } n \leqslant i. \end{cases}$$

In the following, we show that for a fixed n, the existence of a certain $\mathfrak{a}$-filter regular sequence on M characterizes the finiteness properties of $\operatorname{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$.

Theorem 2.4. *For $n \in \mathbb{N}$, the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, H^n_{\mathfrak{a}}(M))$ is finitely generated* *if and only if there exists an* $\mathfrak{a}$*-filter regular sequence* $x_1, \ldots, x_n$ *on* M *such that*

$$\sum_{j=1}^{n} x_j^t M :_M \mathfrak{a} \subseteq \Big(\bigcup_{s \in \mathbb{N}} \Big(\sum_{j=1}^{n} x_j^{t+s} M \Big) :_M x_1^s \cdots x_n^s \Big) + x_1^{t-1} \cdots x_n^{t-1} \Big(\sum_{j=1}^{n} x_j M :_M \mathfrak{a} \Big)$$

for all $t \in \mathbb{N}$.

Proof. Suppose that $y_1, \ldots, y_{n+1} \in \mathfrak{a}$ is an $\mathfrak{a}$-filter regular sequence on M. Then in view of [1, Remark 2.2.17] and Proposition 2.3, there exists the following exact sequence:

$$0 \longrightarrow H_{\mathfrak{a}}^n(M) \longrightarrow H_{(y_1,\ldots,y_n)}^n(M) \longrightarrow (H_{(y_1,\ldots,y_n)}^n(M))_{y_{n+1}}.$$

Since the multiplication by y_{n+1} provides an automorphism on $(H_{(y_1,\ldots,y_n)}^n(M))_{y_{n+1}}$ and $y_{n+1} \in \mathfrak{a}$, by applying the functor $\mathrm{Hom}_R(R/\mathfrak{a}, -)$ on the above exact sequence, we obtain the isomorphism

$$\mathrm{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^n(M)) \cong \mathrm{Hom}_R\big(R/\mathfrak{a},\, H_{(y_1,\ldots,y_n)}^n(M)\big).$$

Let $\mathrm{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^n(M))$ be finitely generated. Then $\mathrm{Hom}_R(R/\mathfrak{a}, H_{(y_1,\ldots,y_n)}^n(M))$ is also finitely generated. Thus, there exists a positive integer u such that the map

$$\varphi_{\mathfrak{a},y^u} : \mathrm{Hom}_R\Big(R/\mathfrak{a},\, \frac{M}{(y_1^u, \ldots, y_n^u)M}\Big) \longrightarrow \mathrm{Hom}_R\big(R/\mathfrak{a},\, H_{(y_1,\ldots,y_n)}^n(M)\big)$$

is surjective, where $y^u := y_1^u, \ldots, y_n^u$. Set $x_i := y_i^u$ for all i with $1 \leqslant i \leqslant n$ and note that $x = x_1, \ldots, x_n$ is again an $\mathfrak{a}$-filter regular sequence on M. Moreover, by [1, Remark 1.2.3], the map

$$\varphi_{\mathfrak{a},x} : \mathrm{Hom}_R\Big(R/\mathfrak{a},\, \frac{M}{(x_1, \ldots, x_n)M}\Big) \longrightarrow \mathrm{Hom}_R\big(R/\mathfrak{a},\, H_{(x_1,\ldots,x_n)}^n(M)\big)$$

is surjective. The result now follows from Lemma 2.1.

Now by employing a method which is similar to that we used in the second paragraph in the present proof, in conjunction with Lemma 2.1, one can complete the proof. $\square$

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