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Analytical solutions for bending analysis of rectangular laminated plates with arbitrary lamination and boundary conditions †

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Abstract

The intent of the present study is to employ the extended Kantorovich method for semi-analytical solutions of laminated composite plates with arbitrary lamination and boundary conditions subjected to transverse loads. The method based on separation of spatial variables of displacement field components. Within the displacement field of a first-order shear deformation theory, a laminated plate theory is developed. Using the principle of minimum total potential energy, two systems of coupled ordinary differential equations with constant coefficients are obtained. The equations are solved analytically by using the state-space approach. The results obtained are compared with the Levy-type solutions of cross-ply and antisymmetric angle-ply laminates with various admissible boundary conditions to verify the validity and accuracy of the present theory. Also, for other laminations and boundary conditions that there exist no Levy-type solutions the present results are compared with those obtained by other investigators and finite element method. It is found that the present results have very good agreements with those obtained by other methods.

Keywords: Extended Kantorovich method; Laminated composite plates; Arbitrary boundary conditions; First-order shear deformation theory

1. Introduction

In recent decades, with the increasing application of composite laminated plates, the interest of more researchers has been attracted to the analysis of this kind of structures. Through the works presented in this field, notably large numbers of them are devoted to analytical methods for bending and free vibration analysis (see, for example, [1-4]). However, most of the methods introduced in these works are restricted to analysis of laminated plates with simply supported edges and especially laminations (i.e., cross-ply and antisymmetric angle-ply laminates).

Up to now, only few theories have been presented that can analyze plates with more general laminations or boundary conditions. In fact, it can be said that hitherto the most popular analytical method for analysis of non-simply supported laminated plates has been the Levy method, which is able to analyze cross-ply and antisymmetric angle-ply laminates with two simply supported opposite edges. Numerous investigators have used the Levy method to solve the governing equations of various equivalent single-layer plate theories (e.g., see [5-9]).

Employing the Levy solutions and method of superposition, Timoshenko [10] studied deflection and bending moments for rectangular isotropic thin plates with all edges clamped, one edge or two adjacent edges simply supported and the other edges clamped, and one edge free and the others clamped. Taking the idea of Timoshenko [10], Bhaskar and Kaushik [11] presented an exact solution for symmetric cross-ply thin plates with any combination of simply supported and clamped edges. Their methodology was based on superposition of the Navier solution corresponding to

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the applied transverse load and a number of double sine series solutions, equal to the number of clamped edges, each corresponding to the appropriate edge moment. Bhaskar and Kaushik [12] developed their previous work for analysis of unsymmetric cross-ply plates with clamped edges. They used the following superposition to produce the clamped boundary conditions on the edges of a simply supported plate: 1) the applied load, 2) bending moments and in-plane normal forces applied along a pair of opposite edges, and 3) similar edge moments and forces acting on the other pair of opposite edges. Similarly, Umasree and Bhaskar [13, 14] studied symmetric cross-ply clamped laminated plates using first-order shear deformation theory (FSDT) and zig–zag type higher-order theory, respectively.

Green [15] introduced a mathematical approach to obtain derivatives of a function represented by a Fourier series that violates the boundary conditions, and hence one that cannot be differentiated further term-by-term. By starting with a series solution for the displacement function as for a simply supported plate, and by obtaining derivatives of this series using Green's methodology, the problem of the plate with other boundary conditions may be reduced to a set of infinite equations in infinite unknowns. The series solution can be appropriately truncated depending on the degree of accuracy desired. Whitney [16] and Kabir and Chaudhuri [17] extended the methodology of Green [15] for thin symmetrically anisotropic and moderately thick cross-ply clamped laminated plates, respectively. Chaudhuri and Kabir [18] presented a similar work on transversely isotropic Mindlin plates. Also, recently Khalili et al. [19] used the described methodology for static and dynamic analysis of symmetric cross-ply laminated plates with different boundary conditions. They had to fulfill an elaborate mathematical procedure to obtain the unknown due to every set of boundary conditions on the edges of the plate. Kabir and Chaudhuri [20] reported a minor variant of Green's approach wherein the assumed displacement functions satisfy the clamped boundary conditions a priori; expansion of cosine functions in a sine series, or vice versa, as suggested by Green and Hearmon [21]. Chaudhuri and Kabir [22] extended their earlier work to derive a boundary-continuous-displacement solution for an arbitrarily laminated clamped plate. They used FSDT and illustrated their results for a general laminate of [0°/60°] construction. A disadvantage of Green's approach, besides the uncertain nature of convergence of the series employed, is the larger number of unknown variables that one has to solve for – namely, the Fourier coefficients of the double series assumed for the displacements as against the coefficients of the single series assumed for the edge moments in the superposition approach.

Vel and Batra [23] generalized the Eshelby–Stroh formalism [24] to study the three-dimensional deformations of anisotropic laminated rectangular plates subjected to arbitrary boundary conditions at the edges. They satisfied the interface continuity and the boundary conditions in the sense of Fourier series, which results in an infinite system of equations in infinite unknowns. The truncation of this set of equations inevitably involves some errors which can be minimized by increasing the number of terms in the series. However, Vel and Batra [23] presented the numerical results, only, for a cross-ply plate simply supported on two opposite edges and subjected to different sets of boundary conditions on the other edges and a clamped plate with [0°/90°/0°] and [45°/-45°/45°] laminations. Vel and Batra [25] simplified three-dimensional equations of linear elasticity to the case of generalized plane-strain deformations and solved them by the Eshelby-Stroh formalism to study the cylindrical bending of an anisotropic laminated plate with either both edges clamped or one edge clamped and the other simply supported or one edge clamped and the other free.

Our purpose is to develop the extended Kantorovich method (EKM) [26] for bending analysis of laminated composite plates with arbitrary lamination and boundary conditions. With the extended Kantorovich approach, it is assumed that a solution is in the form of either a product of two independent functions of problem spatial variables (e.g., $f(x)$ and $g(y)$ for a rectangular plate or $f(x)$ and $g(\theta)$ for a cylindrical panel) or a sum of products of independent functions of problem spatial variables. Taking this assumption along with an energy method, two coupled sets of ordinary differential equations, instead of one set of partial differential equations, are obtained. The coupled differential equations are solved in an iterative manner that starts by guessing a solution for a set of the equations. Then the solution of the other set may be derived analytically or numerically. Subsequently, the obtained solution is used as a beginning point to solve the former set of the equations. This iterative procedure continues until the solution is converged. A few papers have been published on bending analysis of anisotropic plates and shells by the use of the EKM. Dalaei and Kerr [27] utilized the EKM to generate a closed form approximate solution for the deflections of a clamped rectangular orthotropic plate. They showed that the convergence of the method is very rapid, in that the final form of their single-term solution is reached after only four iterations, and that the final results are independent of the initial choice. Aghdam and Falahatgar [28] presented a single-term solution for bending analysis of moderately thick rectangular symmetric cross-ply laminates with clamped edges subjected to a uniform distributed load. Recently, Abouhamze et al. [29] presented a similar work for thin symmetric cross-ply cylindrical panels with clamped edges subjected to three different loading conditions (uniform, linear and sinusoidal). In this study the multi-term version of EKM is conjugated to FSDT for static analysis of rectangular laminated plates with arbitrary boundary conditions subjected to transverse loadings. However, since the procedure used is simple and straightforward, it can be adopted in developing higher-order shear deformation and layerwise laminated plate theories. To check the validity and accuracy of the present method, three numerical examples are presented. The first contains an antisymmetric angle-ply laminated plate which has a Levy-type solution, the second concerns a cross-ply laminate, and the third contains a general laminated plate with arbitrary laminations and boundary conditions. The comparison of the results with those obtained from the other methods shows the excellent accuracy of the present method.

2. Formulation

2.1 Displacement field and strains

Consider a generally laminated plate as shown in Fig. 1 with a total thickness h, width b in the lateral (y-) direction, and length a in the longitudinal (x-) direction. It is assumed that the middle plane of the plate lies on the x-y plane of a Cartesian coordinate system. Here, in order to introduce the idea, the theory will be developed within the framework of the first-order shear deformation theory [30], although the method is general and can be used within any shear deformation plate and shell theories. To this end, it is assumed that the displacement field of the plate may be presented as:

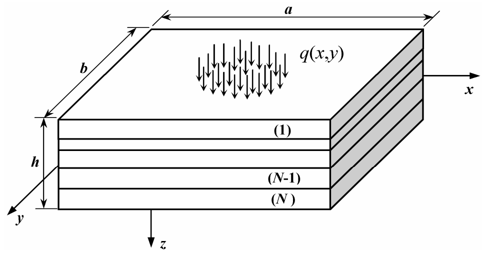


Fig. 1. The plate geometry and coordinate system.

$$u(x,y,z) = u_i(x)\bar{u}_i(y) + z\psi_i(x)\bar{\psi}_i(y)$$
$$v(x,y,z) = v_i(x)\bar{v}_i(y) + z\phi_i(x)\bar{\phi}_i(y)$$
$$w(x,y,z) = w_i(x)\bar{w}_i(y) \quad i = 1,2,\ldots,n \tag{1}$$

where, for the sake of brevity, the Einstein summation convention has been introduced – a repeated index indicates summation over all values of that index. In Eqs. (1) u (x,y,z), v (x,y,z), and w (x,y,z) represent the displacement components in the x, y, and z directions, respectively, of a material point initially located at (x,y,z) in the undeformed laminate. Also, $u_i(x)\bar{u}_i(y)$, $v_i(x)\bar{v}_i(y)$, and $w_i(x)\bar{w}_i(y)$ denote the displacement of a point on the middle plane of the laminate along the x, y, and z directions, respectively, $\psi_i(x)\bar{\psi}_i(y)$ and $\phi_i(x)\bar{\phi}_i(y)$ are the rotations of a transverse normal about the y and x axes, respectively, and n is the total number of terms considered in the summation. When in the formulation process we take $n \geq 2$, the method is referred to as the multi-term Kantorovich approach and otherwise, it is termed the single-term Kantorovich method.

Upon substitution of the displacement field (1) into the linear strain-displacement relations of elasticity [31] the following strain-displacement relations will be obtained:

$$\varepsilon_x = \varepsilon_x^0 + z\kappa_x, \quad \varepsilon_y = \varepsilon_y^0 + z\kappa_y, \quad \varepsilon_z = 0$$
$$\gamma_{yz} = \gamma_{yz}^0, \quad \gamma_{xz} = \gamma_{xz}^0, \quad \gamma_{xy} = \gamma_{xy}^0 + z\kappa_{xy} \tag{2}$$

where

$$\varepsilon_x^0 = u_i\bar{u}_i', \quad \varepsilon_y^0 = v_i\bar{v}_i'$$
$$\kappa_x = \psi_i\bar{\psi}_i', \quad \kappa_y = \phi_i\bar{\phi}_i'$$
$$\gamma_{yz}^0 = \phi_i\bar{\phi}_i + w_i\bar{w}_i', \quad \gamma_{xz}^0 = \psi_i\bar{\psi}_i + w_i'\bar{w}_i$$
$$\gamma_{xy}^0 = u_i\bar{u}_i' + v_i'\bar{v}_i, \quad \kappa_{xy} = \psi_i\bar{\psi}_i' + \phi_i'\bar{\phi}_i \tag{3}$$

In Eqs. (3) a prime indicates an ordinary derivative with respect to the appropriate variable x or y.

2.2 Equilibrium equations

Using the principle of minimum total potential energy [31],

$$\delta U + \delta V = 0 \tag{4}$$

which states that summation of the first variation of internal strain energy and the first variation of the potential of external loads equals zero; two sets of equilibrium equations and boundary conditions are obtained. If the functions $\bar{u}_i$, $\bar{v}_i$, $\bar{w}_i$, $\bar{\psi}_i$, and $\bar{\phi}_i$ are assumed to be known, the first set of equilibrium equations can be shown to be:

$$\begin{aligned}
&\delta u_i: \quad \frac{d\mathcal{N}_x^i}{dx} - \mathcal{N}_{xy1}^i = 0 \\
&\delta v_i: \quad \frac{d\mathcal{N}_{xy2}^i}{dx} - \mathcal{N}_y^i = 0 \\
&\delta \psi_i: \quad \frac{d\mathcal{M}_x^i}{dx} - \mathcal{M}_{xy1}^i - \mathcal{Q}_{x1}^i = 0 \\
&\delta \phi_i: \quad \frac{d\mathcal{M}_{xy2}^i}{dx} - \mathcal{M}_y^i - \mathcal{Q}_{y1}^i = 0 \\
&\delta w_i: \quad \frac{d\mathcal{Q}_{x2}^i}{dx} - \mathcal{Q}_{y2}^i + q_i(x) = 0 \qquad i = 1,2,\ldots,n
\end{aligned} \tag{5}$$

where the generalized stress and moment resultants are defined as:

$$\begin{bmatrix} \{\mathcal{N}^i\}^T \\ \{\mathcal{M}^i\}^T \\ \{\mathcal{Q}^i\}^T \end{bmatrix} = \begin{bmatrix} \mathcal{N}_x^i & \mathcal{N}_y^i & \mathcal{N}_{xy1}^i & \mathcal{N}_{xy2}^i \\ \mathcal{M}_x^i & \mathcal{M}_y^i & \mathcal{M}_{xy1}^i & \mathcal{M}_{xy2}^i \\ \mathcal{Q}_{y1}^i & \mathcal{Q}_{y2}^i & \mathcal{Q}_{x1}^i & \mathcal{Q}_{x2}^i \end{bmatrix} = \int_0^b \begin{bmatrix} N_x\bar{u}_i & N_y\bar{v}_i' & N_{xy}\bar{u}_i' & N_{xy}\bar{v}_i \\ M_x\bar{\psi}_i & M_y\bar{\phi}_i' & M_{xy}\bar{\psi}_i' & M_{xy}\bar{\phi}_i \\ Q_y\bar{\phi}_i & Q_y\bar{w}_i' & Q_x\bar{\psi}_i & Q_x\bar{w}_i \end{bmatrix} dy \tag{6}$$

with

$$q_i(x) = \int_0^b q(x,y)\,\bar{w}_i\,dy \tag{7}$$

where $q\ (x,y)$ is the applied transverse load at z=-h/2. In Eqs. (6) the stress and moment resultants are:

$$\begin{aligned}
(N_x, N_y, N_{xy}, Q_y, Q_x) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yz}, \sigma_{xz})\,dz \\
(M_x, M_y, M_{xy}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy})\,z\,dz
\end{aligned} \tag{8}$$

The boundary conditions corresponding to Eqs. (5) consist of specifying the following quantities at the edges parallel to the y-axis (i.e., at x=0,a):

Primary variables: u_i, v_i, ψ_i, ϕ_i, w_i

Secondary variables: $\mathcal{N}_x^i$, $\mathcal{N}_{xy2}^i$, $\mathcal{M}_x^i$, $\mathcal{M}_{xy2}^i$, $\mathcal{Q}_{x2}^i$ i=1,2,…,n (9)

If, on the other hand, the functions u_i, v_i, w_i, ψ_i, and ϕ_i are assumed to be known, then the second set of equilibrium equations will be:

$$\begin{aligned}
&\delta \bar{u}_i: \quad \frac{d\bar{\mathcal{N}}_{xy1}^i}{dy} - \bar{\mathcal{N}}_x^i = 0 \\
&\delta \bar{v}_i: \quad \frac{d\bar{\mathcal{N}}_y^i}{dy} - \bar{\mathcal{N}}_{xy2}^i = 0 \\
&\delta \bar{\psi}_i: \quad \frac{d\bar{\mathcal{M}}_{xy1}^i}{dy} - \bar{\mathcal{M}}_x^i - \bar{\mathcal{Q}}_{x1}^i = 0 \\
&\delta \bar{\phi}_i: \quad \frac{d\bar{\mathcal{M}}_y^i}{dy} - \bar{\mathcal{M}}_{xy2}^i - \bar{\mathcal{Q}}_{y1}^i = 0 \\
&\delta \bar{w}_i: \quad \frac{d\bar{\mathcal{Q}}_{y2}^i}{dy} - \bar{\mathcal{Q}}_{x2}^i + \bar{q}_i(y) = 0 \qquad i = 1,2,\ldots,n
\end{aligned} \tag{10}$$

In the above equations the generalized stress and moment resultants are defined as:

$$\begin{bmatrix} \{\bar{\mathcal{N}}^i\}^T \\ \{\bar{\mathcal{M}}^i\}^T \\ \{\bar{\mathcal{Q}}^i\}^T \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{N}}_x^i & \bar{\mathcal{N}}_y^i & \bar{\mathcal{N}}_{xy1}^i & \bar{\mathcal{N}}_{xy2}^i \\ \bar{\mathcal{M}}_x^i & \bar{\mathcal{M}}_y^i & \bar{\mathcal{M}}_{xy1}^i & \bar{\mathcal{M}}_{xy2}^i \\ \bar{\mathcal{Q}}_{y1}^i & \bar{\mathcal{Q}}_{y2}^i & \bar{\mathcal{Q}}_{x1}^i & \bar{\mathcal{Q}}_{x2}^i \end{bmatrix} = \int_0^a \begin{bmatrix} N_x u_i' & N_y v_i & N_{xy} u_i & N_{xy} v_i' \\ M_x \psi_i' & M_y \phi_i & M_{xy}\psi_i & M_{xy}\phi_i' \\ Q_y\phi_i & Q_y w_i & Q_x\psi_i & Q_x w_i' \end{bmatrix} dx \tag{11}$$

with

$$\bar{q}_i(y) = \int_0^a q(x,y)\,w_i\,dy \tag{12}$$

The boundary conditions corresponding to Eqs. (10) consist of specifying the following quantities at the edges parallel to the x-axis (i.e., at y=0,b):

Primary variables: $\bar{u}_i$, $\bar{v}_i$, $\bar{\psi}_i$, $\bar{\phi}_i$, $\bar{w}_i$

Secondary variables: $\bar{\mathcal{N}}_{xy1}^i$, $\bar{\mathcal{N}}_y^i$, $\bar{\mathcal{M}}_{xy1}^i$, $\bar{\mathcal{M}}_y^i$, $\bar{\mathcal{Q}}_{y2}^i$ i=1,2,…,n (13)

2.3 Laminate constitutive relations

The linear plane stress constitutive relations for the kth orthotropic lamina with respect to the laminate coordinate axes (see Fig. 1) are given by [30]:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^{(k)}$$

$$\begin{Bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^{(k)} \quad (14)$$

where $[\bar{Q}]^{(k)}$ is the transformed reduced stiffness matrix and $\bar{C}_{ij}^{(k)}$ (i,j=4,5) are the off-axis stiffness coefficients of the kth lamina. Upon substitution of Eqs. (2) into Eqs. (14) and the subsequent results into Eqs. (8), the stress and moment resultants are obtained, which can be presented as follows:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ & & A_{66} & B_{16} & B_{26} & B_{66} \\ & & & D_{11} & D_{12} & D_{16} \\ & Sym. & & & D_{22} & D_{26} \\ & & & & & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = k^2 \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} \quad (15)$$

Here, k^2 (=5/6) is the shear correction factor introduced as in the first-order shear deformation plate and shell theories. Also A_{ij}, B_{ij}, and D_{ij} (i,j=1,2,6) denote the extensional stiffnesses, the bending-extensional coupling stiffnesses, and the bending stiffnesses, respectively. These stiffnesses are given by:

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)} (1, z, z^2) dz$$
$$i, j = 1,2,6$$
$$A_{ij} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{C}_{ij}^{(k)} dz \quad i, j = 4,5 \quad (16)$$

where N is the total number of layers. Upon substitution of Eqs. (2) into (15) and the subsequent results into Eqs. (6) and (11), the generalized stress resultants are obtained, which can be represented as follows:

$$\begin{Bmatrix} \{\mathcal{N}^i\} \\ \{\mathcal{M}^i\} \end{Bmatrix} = [\mathcal{A}^{ij}]\{\xi_j\}$$
$$\{\mathcal{Q}^i\} = [\mathcal{B}^{ij}]\{\eta_j\} \quad i, j = 1,2,\ldots,n \quad (17)$$

$$\begin{Bmatrix} \{\bar{\mathcal{N}}^i\} \\ \{\bar{\mathcal{M}}^i\} \end{Bmatrix} = [\bar{\mathcal{A}}^{ij}]\{\bar{\xi}_j\}$$
$$\{\bar{\mathcal{Q}}^i\} = [\bar{\mathcal{B}}^{ij}]\{\bar{\eta}_j\} \quad i, j = 1,2,\ldots,n \quad (18)$$

where

$$\{\xi_j\} = \begin{bmatrix} u'_j & v_j & u_j & v'_j & \psi'_j & \phi_j & \psi_j & \phi'_j \end{bmatrix}^T$$
$$\{\eta_j\} = \begin{bmatrix} \phi_j & w_j & \psi_j & w'_j \end{bmatrix}^T \quad (19)$$
$$\{\bar{\xi}_j\} = \begin{bmatrix} \bar{u}_j & \bar{v}'_j & \bar{u}'_j & \bar{v}_j & \bar{\psi}_j & \bar{\phi}'_j & \bar{\psi}'_j & \bar{\phi}_j \end{bmatrix}^T$$
$$\{\bar{\eta}_j\} = \begin{bmatrix} \bar{\phi}_j & \bar{w}'_j & \bar{\psi}_j & \bar{w}_j \end{bmatrix}^T \quad (20)$$

and the coefficient matrices $[\mathcal{A}^{ij}]$, $[\mathcal{B}^{ij}]$, $[\bar{\mathcal{A}}^{ij}]$, and $[\bar{\mathcal{B}}^{ij}]$ in Eqs. (17) and (18) are defined as:

$$[\mathcal{A}^{ij}] = \int_0^b \left([\alpha] \otimes \{\bar{\xi}_i\}\{\bar{\xi}_j\}^T \right) dy$$
$$[\mathcal{B}^{ij}] = \int_0^b \left([\beta] \otimes \{\bar{\eta}_i\}\{\bar{\eta}_j\}^T \right) dy \quad (21)$$
$$[\bar{\mathcal{A}}^{ij}] = \int_0^a \left([\alpha] \otimes \{\xi_i\}\{\xi_j\}^T \right) dx$$
$$[\bar{\mathcal{B}}^{ij}] = \int_0^a \left([\beta] \otimes \{\eta_i\}\{\eta_j\}^T \right) dx \quad (22)$$

In Eqs. (21) and (22) $[\alpha]$ and $[\beta]$ are:

$$[\alpha] = \begin{bmatrix} A_{11} & A_{12} & A_{16} & A_{16} & B_{11} & B_{12} & B_{16} & B_{16} \\ & A_{22} & A_{26} & A_{26} & B_{12} & B_{22} & B_{26} & B_{26} \\ & & A_{66} & A_{66} & B_{16} & B_{26} & B_{66} & B_{66} \\ & & & A_{66} & B_{16} & B_{26} & B_{66} & B_{66} \\ & & & & D_{11} & D_{12} & D_{16} & D_{16} \\ & & Sym. & & & D_{22} & D_{26} & D_{26} \\ & & & & & & D_{66} & D_{66} \\ & & & & & & & D_{66} \end{bmatrix} \quad (23)$$

$$[\beta] = k^2 \begin{bmatrix} A_{44} & A_{44} & A_{45} & A_{45} \\ & A_{44} & A_{45} & A_{45} \\ & & A_{55} & A_{55} \\ Sym. & & & A_{55} \end{bmatrix} \quad (24)$$

It must be noted that the sign ⊗ used in Eqs. (21) and (22) is referred to as *array multiplication* of two matrices.

2.4 Governing equations of equilibrium

The equilibrium equations in (4) and (9) can be expressed in terms of displacements by substituting the generalized stress resultants from (16) and (17). Hence, two sets of ordinary differential equations with constant coefficients will be obtained as follows:

$$\begin{aligned}
\delta u_i :\ & \mathcal{A}_{11}^{ij}u_j'' + (\mathcal{A}_{13}^{ij} - \mathcal{A}_{31}^{ij})u_j' - \mathcal{A}_{33}^{ij}u_j + \mathcal{A}_{14}^{ij}v_j'' + (\mathcal{A}_{12}^{ij} \\
& - \mathcal{A}_{34}^{ij})v_j' - \mathcal{A}_{32}^{ij}v_j + \mathcal{A}_{15}^{ij}\psi_j'' + (\mathcal{A}_{17}^{ij} - \mathcal{A}_{35}^{ij})\psi_j' \\
& - \mathcal{A}_{37}^{ij}\psi_j + \mathcal{A}_{18}^{ij}\phi_j'' + (\mathcal{A}_{16}^{ij} - \mathcal{A}_{38}^{ij})\phi_j' - \mathcal{A}_{36}^{ij}\phi_j = 0 \\
\delta v_i :\ & \mathcal{A}_{41}^{ij}u_j'' + (\mathcal{A}_{43}^{ij} - \mathcal{A}_{21}^{ij})u_j' - \mathcal{A}_{23}^{ij}u_j + \mathcal{A}_{44}^{ij}v_j'' + (\mathcal{A}_{42}^{ij} \\
& - \mathcal{A}_{24}^{ij})v_j' - \mathcal{A}_{22}^{ij}v_j + \mathcal{A}_{45}^{ij}\psi_j'' + (\mathcal{A}_{47}^{ij} - \mathcal{A}_{25}^{ij})\psi_j' \\
& - \mathcal{A}_{27}^{ij}\psi_j + \mathcal{A}_{48}^{ij}\phi_j'' + (\mathcal{A}_{46}^{ij} - \mathcal{A}_{28}^{ij})\phi_j' - \mathcal{A}_{26}^{ij}\phi_j = 0 \\
\delta \psi_i :\ & \mathcal{A}_{51}^{ij}u_j'' + (\mathcal{A}_{53}^{ij} - \mathcal{A}_{71}^{ij})u_j' - \mathcal{A}_{73}^{ij}u_j + \mathcal{A}_{54}^{ij}v_j'' + (\mathcal{A}_{52}^{ij} \\
& - \mathcal{A}_{74}^{ij})v_j' - \mathcal{A}_{72}^{ij}v_j + \mathcal{A}_{55}^{ij}\psi_j'' + (\mathcal{A}_{57}^{ij} - \mathcal{A}_{75}^{ij})\psi_j' \\
& - (\mathcal{A}_{77}^{ij} + \mathcal{B}_{33}^{ij})\psi_j + \mathcal{A}_{58}^{ij}\phi_j'' + (\mathcal{A}_{56}^{ij} - \mathcal{A}_{78}^{ij})\phi_j' \\
& - (\mathcal{A}_{76}^{ij} + \mathcal{B}_{31}^{ij})\phi_j - \mathcal{B}_{34}^{ij}w_j' - \mathcal{B}_{32}^{ij}w_j = 0 \\
\delta \phi_i :\ & \mathcal{A}_{81}^{ij}u_j'' + (\mathcal{A}_{83}^{ij} - \mathcal{A}_{61}^{ij})u_j' - \mathcal{A}_{63}^{ij}u_j + \mathcal{A}_{84}^{ij}v_j'' + (\mathcal{A}_{82}^{ij} \\
& - \mathcal{A}_{64}^{ij})v_j' - \mathcal{A}_{62}^{ij}v_j + \mathcal{A}_{85}^{ij}\psi_j'' + (\mathcal{A}_{87}^{ij} - \mathcal{A}_{65}^{ij})\psi_j' \\
& - (\mathcal{A}_{67}^{ij} + \mathcal{B}_{13}^{ij})\psi_j + \mathcal{A}_{88}^{ij}\phi_j'' + (\mathcal{A}_{86}^{ij} - \mathcal{A}_{68}^{ij})\phi_j' \\
& - (\mathcal{A}_{66}^{ij} + \mathcal{B}_{11}^{ij})\phi_j - \mathcal{B}_{14}^{ij}w_j' - \mathcal{B}_{12}^{ij}w_j = 0 \\
\delta w_i :\ & \mathcal{B}_{43}^{ij}\psi_j' - \mathcal{B}_{23}^{ij}\psi_j + \mathcal{B}_{41}^{ij}\phi_j' - \mathcal{B}_{21}^{ij}\phi_j + \mathcal{B}_{44}^{ij}w_j'' \\
& + (\mathcal{B}_{42}^{ij} - \mathcal{B}_{24}^{ij})w_j' - \mathcal{B}_{22}^{ij}w_j = -q_i(x)
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
\delta \bar{u}_i :\ & \bar{\mathcal{A}}_{33}^{ij}\bar{u}_j'' + (\bar{\mathcal{A}}_{31}^{ij} - \bar{\mathcal{A}}_{13}^{ij})\bar{u}_j' - \bar{\mathcal{A}}_{11}^{ij}\bar{u}_j + \bar{\mathcal{A}}_{32}^{ij}\bar{v}_j'' + (\bar{\mathcal{A}}_{34}^{ij} \\
& - \bar{\mathcal{A}}_{12}^{ij})\bar{v}_j' - \bar{\mathcal{A}}_{14}^{ij}\bar{v}_j + \bar{\mathcal{A}}_{37}^{ij}\bar{\psi}_j'' + (\bar{\mathcal{A}}_{35}^{ij} - \bar{\mathcal{A}}_{17}^{ij})\bar{\psi}_j' \\
& - \bar{\mathcal{A}}_{15}^{ij}\bar{\psi}_j + \bar{\mathcal{A}}_{36}^{ij}\bar{\phi}_j'' + (\bar{\mathcal{A}}_{38}^{ij} - \bar{\mathcal{A}}_{16}^{ij})\bar{\phi}_j' - \bar{\mathcal{A}}_{18}^{ij}\bar{\phi}_j = 0 \\
\delta \bar{v}_i :\ & \bar{\mathcal{A}}_{23}^{ij}\bar{u}_j'' + (\bar{\mathcal{A}}_{21}^{ij} - \bar{\mathcal{A}}_{43}^{ij})\bar{u}_j' - \bar{\mathcal{A}}_{41}^{ij}\bar{u}_j + \bar{\mathcal{A}}_{22}^{ij}\bar{v}_j'' + (\bar{\mathcal{A}}_{24}^{ij} \\
& - \bar{\mathcal{A}}_{42}^{ij})\bar{v}_j' - \bar{\mathcal{A}}_{44}^{ij}\bar{v}_j + \bar{\mathcal{A}}_{27}^{ij}\bar{\psi}_j'' + (\bar{\mathcal{A}}_{25}^{ij} - \bar{\mathcal{A}}_{47}^{ij})\bar{\psi}_j' \\
& - \bar{\mathcal{A}}_{45}^{ij}\bar{\psi}_j + \bar{\mathcal{A}}_{26}^{ij}\bar{\phi}_j'' + (\bar{\mathcal{A}}_{28}^{ij} - \bar{\mathcal{A}}_{46}^{ij})\bar{\phi}_j' - \bar{\mathcal{A}}_{48}^{ij}\bar{\phi}_j = 0 \\
\delta \bar{\psi}_i :\ & \bar{\mathcal{A}}_{73}^{ij}\bar{u}_j'' + (\bar{\mathcal{A}}_{71}^{ij} - \bar{\mathcal{A}}_{53}^{ij})\bar{u}_j' - \bar{\mathcal{A}}_{51}^{ij}\bar{u}_j + \bar{\mathcal{A}}_{72}^{ij}\bar{v}_j'' + (\bar{\mathcal{A}}_{74}^{ij} \\
& - \bar{\mathcal{A}}_{52}^{ij})\bar{v}_j' - \bar{\mathcal{A}}_{54}^{ij}\bar{v}_j + \bar{\mathcal{A}}_{77}^{ij}\bar{\psi}_j'' + (\bar{\mathcal{A}}_{75}^{ij} - \bar{\mathcal{A}}_{57}^{ij})\bar{\psi}_j' \\
& - (\bar{\mathcal{A}}_{55}^{ij} + \bar{\mathcal{B}}_{33}^{ij})\bar{\psi}_j + \bar{\mathcal{A}}_{76}^{ij}\bar{\phi}_j'' + (\bar{\mathcal{A}}_{78}^{ij} - \bar{\mathcal{A}}_{56}^{ij})\bar{\phi}_j' \\
& - (\bar{\mathcal{A}}_{58}^{ij} + \bar{\mathcal{B}}_{31}^{ij})\bar{\phi}_j - \bar{\mathcal{B}}_{32}^{ij}\bar{w}_j' - \bar{\mathcal{B}}_{34}^{ij}\bar{w}_j = 0 \\
\delta \bar{\phi}_i :\ & \bar{\mathcal{A}}_{63}^{ij}\bar{u}_j'' + (\bar{\mathcal{A}}_{61}^{ij} - \bar{\mathcal{A}}_{83}^{ij})\bar{u}_j' - \bar{\mathcal{A}}_{81}^{ij}\bar{u}_j + \bar{\mathcal{A}}_{62}^{ij}\bar{v}_j'' + (\bar{\mathcal{A}}_{64}^{ij} \\
& - \bar{\mathcal{A}}_{82}^{ij})\bar{v}_j' - \bar{\mathcal{A}}_{84}^{ij}\bar{v}_j + \bar{\mathcal{A}}_{67}^{ij}\bar{\psi}_j'' + (\bar{\mathcal{A}}_{65}^{ij} - \bar{\mathcal{A}}_{87}^{ij})\bar{\psi}_j' \\
& - (\bar{\mathcal{A}}_{85}^{ij} + \bar{\mathcal{B}}_{13}^{ij})\bar{\psi}_j + \bar{\mathcal{A}}_{66}^{ij}\bar{\phi}_j'' + (\bar{\mathcal{A}}_{68}^{ij} - \bar{\mathcal{A}}_{86}^{ij})\bar{\phi}_j' \\
& - (\bar{\mathcal{A}}_{88}^{ij} + \bar{\mathcal{B}}_{11}^{ij})\bar{\phi}_j - \bar{\mathcal{B}}_{12}^{ij}\bar{w}_j' - \bar{\mathcal{B}}_{14}^{ij}\bar{w}_j = 0 \\
\delta \bar{w}_i :\ & \bar{\mathcal{B}}_{23}^{ij}\bar{\psi}_j' - \bar{\mathcal{B}}_{43}^{ij}\bar{\psi}_j + \bar{\mathcal{B}}_{21}^{ij}\bar{\phi}_j' - \bar{\mathcal{B}}_{41}^{ij}\bar{\phi}_j + \bar{\mathcal{B}}_{22}^{ij}\bar{w}'' \\
& + (\bar{\mathcal{B}}_{24}^{ij} - \bar{\mathcal{B}}_{42}^{ij})\bar{w}_j' - \bar{\mathcal{B}}_{44}^{ij}\bar{w}_j = -\bar{q}_i(y)
\end{aligned} \tag{26}$$

3. The solution procedure of the equilibrium equations

To solve Eqs. (25), for convenience the following state space vectors are introduced:

$$\begin{aligned}
&\{X_1\} = \{u(x)\}, \quad \{X_2\} = \{u'(x)\}, \quad \{X_3\} = \{v(x)\} \\
&\{X_4\} = \{v'(x)\}, \quad \{X_5\} = \{\psi(x)\}, \quad \{X_6\} = \{\psi'(x)\} \\
&\{X_7\} = \{\phi(x)\}, \quad \{X_8\} = \{\phi'(x)\}, \quad \{X_9\} = \{w(x)\} \\
&\{X_{10}\} = \{w'(x)\}
\end{aligned} \tag{27}$$

Note that in the above relations each of the state space variables, intrinsically, is an $n \times 1$ vector (e.g., $\{\bar{u}'\}^T = [\bar{u}_1' \quad \bar{u}_2' \quad \dots \quad \bar{u}_n']$). Substitution of Eqs. (27) into Eqs. (25) results in a system of five coupled first-order ordinary differential equations which, on the other hand, may be presented as:

$$\{X'\} = [T]\{X\} + \{F\} \tag{28}$$

To solve Eqs. (28), let us assume that $\bar{u}_i(y)$, $\bar{v}_i(y)$,…, and $\bar{w}_i(y)$ are chosen so that the boundary conditions at y=0,b are identically satisfied. Next, the coefficient matrices $[\mathcal{A}^{ij}]$ and $[\mathcal{B}^{ij}]$ are found. Since these coefficients are constant, Eqs. (28) will be five linear ordinary differential equations with constant coefficients. Also, $q_i(x)$ is found from Eq. (7). Now Eqs. (28) may be solved analytically for any boundary conditions at x=0,a to yield the solution of $u_i(x)$, $v_i(x)$,…, and $w_i(x)$. The general solution of Eqs. (28) is given by [32]:

$$\begin{aligned}
\{X\} &= [U][Q(x)]\{K\} \\
&\quad + [U][Q(x)]\int_0^x [Q(\zeta)]^{-1}[U]^{-1}\{F(\zeta)\}\,d\zeta
\end{aligned} \tag{29}$$

where [U] is the matrix of distinct eigenvectors of matrix [T] and {K} is a vector of unknown constants to be found by imposing the boundary conditions at the edges x=0,a. Also, the diagonal matrix [Q] is defined as:

$$[Q] = \mathrm{diag}(e^{\lambda_1 x}, e^{\lambda_2 x}, \dots, e^{\lambda_{10n} x}) \tag{30}$$

where λ_k $(k = 1, 2, \dots, 10n)$ are the eigenvalues of the coefficient matrix [T] which, in general, must be regarded to have complex values.

Next, we can substitute the obtained general solution of $u_i(x)$, $v_i(x)$,…, and $w_i(x)$ into Eqs. (22) to find the coefficient matrices $[\bar{\mathcal{A}}^{ij}]$ and $[\bar{\mathcal{B}}^{ij}]$ which, here, will be constant. The solution procedure for Eqs. (26) is completely analogous to the one presented for Eqs. (25) and therefore, for the sake of brevity will not be taken up here. This procedure (solving the coupled systems of ordinary differential equations) will be continued until the solution is converged. It is to be noted, generally, the initial guesses to start iterative procedure are arbitrary functions and are not required to satisfy any of the boundary conditions. This latitude for selection of initial assumed functions is related to this fact that the boundary conditions are automatically satisfied in the subsequent iterations. Also, the iterative essence of the method requires that for a specified value of n, the preciseness of the converged solution be independent of the form of the initial guesses.

4. Numerical results

Based on the theoretical formulation discussed in the preceding sections, a computer program was provided to solve the bending problems of laminated plates. Also, based on the Levy method and within the framework of FSDT, two other codes were written to analyze general cross-ply and antisymmetric angle-ply laminated plates with two opposite edges simply supported. Three different numerical examples are studied in this section to demonstrate the validity and accuracy of the present method and its capability to analyze laminated plates with various laminations and boundary conditions. The results obtained from this theory are compared with those obtained by the Levy method, for the cases that Levy's solution exists (i.e., for cross-ply and antisymmetric angle-ply laminates with at least two simply supported opposite edges). For other cases that there exist no Levy-type solutions, the present results are compared with those of finite element analysis as well as those presented by Umasree and Bhaskar [13] and Chaudhuri and Kabir [22].

In all examples, each lamina is assumed to be of the same thickness and has the following orthotropic material properties in the principal material coordinate system [30]:

$$E_1 = 25E_2, \quad G_{12} = G_{13} = 0.5E_2$$
$$G_{23} = 0.2E_2, \quad \nu_{12} = 0.25 \tag{31}$$

with $E_2 = 12$ GPa . In Eqs. (31) E, G, and ν denote Young's modulus, shear modulus, and Poisson's ratio, respectively, and the subscripts 1, 2, and 3 indicate the on-axis material coordinates.

Denoting simply supported, clamped and free boundary conditions by S, C, and F, a 4-word notation such as SFSC is employed to show the boundary conditions on the four edges of the plate. The 1-4th word indicates the boundary conditions on edges x=0, y=0, x=a, and y=b respectively. It can be shown that, as far as analytical solution is concerned, Levy's solution exists only for antisymmetric angle-ply and any cross-ply laminated plates if at least two parallel opposite edges of the plate have simple supports. More specifically, for antisymmetric angle-ply laminates the simple support conditions in the first-order shear deformation laminated plate theory must, say at x=0,a, be [30]:

$$\text{S1 type:} \quad u_0 = N_{xy} = M_x = \phi = w = 0 \tag{32}$$

and also for cross-ply laminates the simple support conditions must, say at x=0,a, be [30]:

$$\text{S2 type:} \quad N_x = v_0 = M_x = \phi = w = 0 \tag{33}$$

All the numerical results for displacements and stresses shown in what follows are presented by means of the following non-dimensionalized quantities:

$$(\bar{u},\bar{v}) = (u,v)\left(\frac{E_2h^2}{b^3q_0}\right)\times 10^2, \quad \bar{w} = w\left(\frac{E_2h^3}{b^4q_0}\right)\times 10^2$$
$$(\bar{\sigma}_x,\bar{\sigma}_y,\bar{\sigma}_{xy}) = (\sigma_x,\sigma_y,\sigma_{xy})\left(\frac{h^2}{b^2q_0}\right)$$
$$(\bar{\sigma}_{xz},\bar{\sigma}_{xz}) = (\sigma_{xz},\sigma_{xz})\left(\frac{h}{bq_0}\right), \quad \bar{\sigma}_z = \sigma_z\left(\frac{1}{q_0}\right) \tag{34}$$

where q_0 is the intensity of the applied uniform transverse load or the amplitude of a double-sinusoidal transverse load which is defined as:

$$q = q_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{35}$$

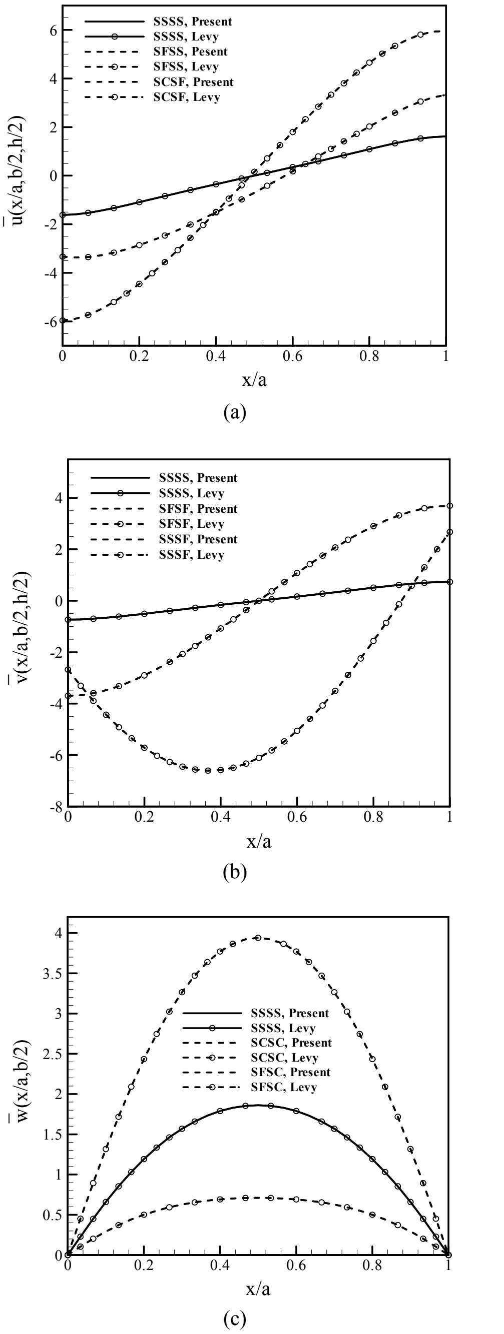


Fig. 2. Variations of the (a) axial, (b) lateral, and (c) transverse displacement vs. x at $y=b/2$ for $[45°/0°/-45°]_2$ laminate compared with the Levy-type solution for various sets of boundary conditions.

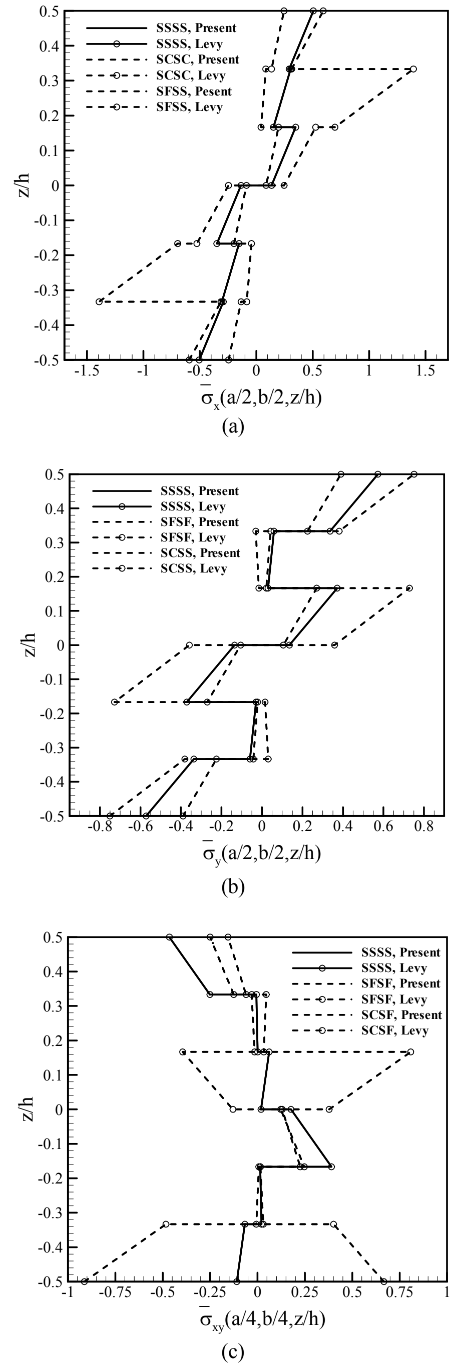


Fig. 3. Distributions of the in-plane stresses through-thickness of $[45°/0°/-45°]_2$ laminate compared to the Levy-type solution for various sets of boundary conditions: (a) σ_x, (b) σ_y, and (c) σ_{xy}.

Example 1

The methodology outlined previously is initially applied to the bi-directional bending problem of a six-layer laminate $[45°/0°/-45°]_2$ subjected to a uniform transverse load q_0. It is assumed that two opposite edges of the laminate have S1-type of simple supports and the other two edges can each be free, simply supported, or clamped, independent of the other. It is also assumed that the plate has length-to-width ratio a/b=2 and width-to-thickness ratio b/h=25.

Figs. 2(a) and 2(b) show, respectively, the variations of displacements $\bar{u}$ and $\bar{v}$ in the bottom surface of the laminate, versus x at $y=b/2$ for several sets of boundary conditions. The variations of transverse deflection along the longitudinal centerline of the laminated plate, corresponding to three sets of boundary conditions (i.e., SSSS, SCSC, and SFSC), are displayed in Fig. 2(c). Also, Figs. 3 and 4 illustrate the through-thickness distributions of the in-plane and interlaminar stresses, respectively. It is to be noted that the interlaminar stresses in this example and the subsequent examples are determined by integrating the local equilibrium equations of the three dimensional elasticity, instead of computing them directly from the constitutive equations (Eqs. 14). The above-mentioned figures indicate that there is an excellent agreement between the results of the present method with those obtained by the Levy method as it is difficult to distinguish the curves of the present method from the curves of the Levy-type method, for a specific boundary conditions.

Example 2

Bending behavior of a symmetric cross-ply square laminate $[0°/90°]_s$ clamped or simply supported along all its edges was studied by Umasree and Bhaskar [13] through an analytical approach. Table 1 compares the results of the present method with those given by Umasree and Bhaskar [13] for the clamped laminated plate subjected to a uniform pressure on its top surface. It is clearly seen that for all the aspect ratios studied, the results of the both methods match very well and variations of the aspect ratios do not have a significant influence on the accuracy of the results. It is noted that the numerical values of σ_y and σ_{xy} tabled in this example are computed in the 90º layer.

Bending of a $[0°/90°]_s$ laminate with various Levy's admissible boundary conditions, under uniform and double-sinusoidal distributed loads is also examined, whose results for a/h=50 are listed in Tables 2 and 3, respectively. It is to be noted that all the simple supports must be assumed to be of type S2. It is seen from these tables that there is an outstanding agreement between the results of the present method and those of the Levy method and Umasree and Bhaskar [13]; however, the precision of the results is affected by the kind of boundary conditions imposed on

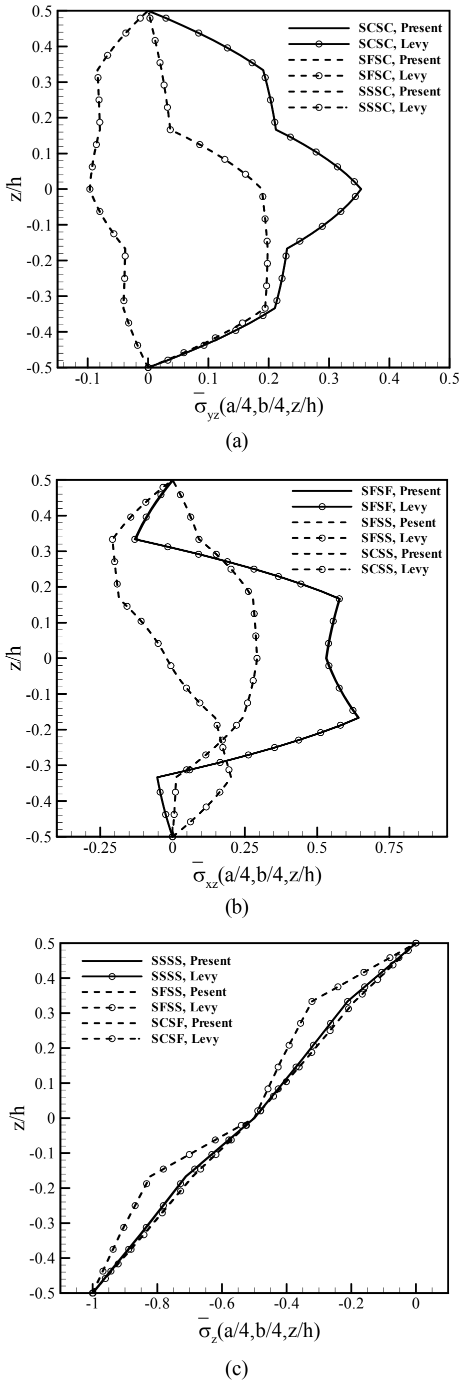


Fig. 4. Distributions of the interlaminar stresses through-thickness of $[45°/0°/-45°]_2$ laminate compared to the Levy-type solution for various sets of boundary conditions: (a) σ_{yz}, (b) σ_{xz}, and (c) σ_z.

the edges of the plate. Also, the results presented in Tables 2 and 3 generally indicate that in the double-sinusoidal loading cases the results obtained by the present method and the Levy method have a better congruity compared to the uniform loading cases.

Table 4 shows the influence of the numerical value of n (the total number of summed terms in Eqs. (1)) on the accuracy of the numerical results obtained from the present method. The numerical values of displacements and stresses listed in Table 4 have been computed for the clamped $[0°/90°]_s$ laminated plate with aspect ratio of a/h=50, under a uniformly distributed load. The present results are compared with those given by Umasree and Bhaskar [13]. These results indicate that as the number n is increased, the accuracy of the results is also increased. However, it is observed that the rate of convergence of stresses is slower compared with that of displacements. The displacements converge to four decimal places with n=2 while some stress components need n=5 for this convergence.

Example 3

Finally, consider a [0°/60°] square laminated

Table 1. Results of the present method and Umasree and Bhaskar [13] for clamped $[0°/90°]_s$ square plate under a uniform pressure.

Method	$\frac{a}{h}$	$\overline{w}\left(\frac{a}{2},\frac{a}{2}\right)$	$\overline{\sigma}_x\left(\frac{a}{2},\frac{a}{2},\frac{h}{2}\right)$	$\overline{\sigma}_y\left(\frac{a}{2},\frac{a}{2},\frac{h}{4}\right)$	$\overline{\sigma}_{xy}\left(\frac{a}{4},\frac{a}{4},\frac{h}{4}\right)$
Present method	10	0.4650	0.2250	0.2571	-0.0034
Umasree and Bhaskar [13]		0.4651	0.2251	0.2572	-0.0034
Present method	20	0.2342	0.2648	0.1623	-0.0032
Umasree and Bhaskar [13]		0.2342	0.2649	0.1623	-0.0032
Present method	50	0.1590	0.2848	0.1113	-0.0031
Umasree and Bhaskar [13]		0.1590	0.2848	0.1113	-0.0031
Present method	100	0.1475	0.2880	0.1012	-0.0031
Umasree and Bhaskar [13]		0.1475	0.2880	0.1012	-0.0030

Table 2. Results of the present method, the Levy method, and Umasree and Bhaskar [13] for $[0°/90°]_s$ square plate under a uniform pressure (a/h=50).

Method		SSSS	CSCS	FSFS	CSFS	CSSS	FSSS
Present method	$\overline{w}(a/2,a/2)$	0.6833	0.1614	3.8874	1.1861	0.3105	2.6729
Levy's method		0.6833	0.1614	3.8876	1.1859	0.3105	2.6729
Umasree and Bhaskar [13]		0.6833	-	-	-	-	-
Present method	$\overline{\sigma}_x(a/2,a/2,h/2)$	0.8228	0.3165	-0.0004	-0.2102	0.4634	0.3094
Levy's method		0.8228	0.3166	-0.0004	-0.2112	0.4634	0.3091
Umasree and Bhaskar [13]		0.8228	-	-	-	-	-
Present method	$\overline{\sigma}_y(a/2,a/2,h/4)$	0.3560	0.0456	2.3356	0.6723	0.1312	1.5851
Levy's method		0.3560	0.0457	2.3358	0.6709	0.1313	1.5848
Umasree and Bhaskar [13]		0.3559	-	-	-	-	-
Present method	$\overline{\sigma}_{xy}(a/4,a/4,h/4)$	-0.0078	-0.0021	0.0005	-0.0141	-0.0041	0.0243
Levy's method		-0.0079	-0.0021	0.0005	-0.0141	-0.0041	0.0243
Umasree and Bhaskar [13]		-0.0079	-	-	-	-	-
Present method	$\overline{\sigma}_{yz}(a/4,a/4,0)$	0.0303	-0.0345	0.6550	-0.0027	-0.0263	0.6395
Levy's method		0.0299	-0.0352	0.6550	-0.0029	-0.0269	0.6390
Present method	$\overline{\sigma}_{xz}(a/4,a/4,0)$	0.2346	0.2966	0.0020	0.4981	0.4138	0.0621
Levy's method		0.2347	0.2968	0.0019	0.5008	0.4137	0.0622
Present method	$\overline{\sigma}_z(a/4,a/4,0)$	-0.5002	-0.5005	-0.5013	-0.5041	-0.5004	-0.4990
Levy's method		-0.5005	-0.5005	-0.5075	-0.5003	-0.5005	-0.5005

plate with aspect ratio of 10, subjected to uniformly distributed loading. As it is evident, the Levy method cannot be applied to obtain solutions for such a lamination configuration. Here, the numerical results, obtained by using the proposed analytical method, are presented for such a laminated plate with arbitrary combination of boundary conditions. The variations of displacement components along the longitudinal centerline of the plate are displayed in Fig. 5. Also, Figures 6 and 7 show the distributions of in-plane and interlaminar stresses through-thickness of the plate, respectively. Although there is no restriction, all simple supports in this example are assumed to be of type S1 for the sake of convenience.

To assess the validity of the results, they are compared with those obtained by employing the commercial finite element package of ANSYS. It is to be noted that the laminated plate has been modeled in ANSYS by using three-dimensional 20-node layered structural solid elements which is proper to model thick laminates. What has been shown through Figures 5, 6 and 7 indicates that the agreement between the results of the present method and finite element the results of the present method and finite element method (FEM) is quite good. The discrepancy between the present results and those by FEM can be referred to the fact that the plate has been considered moderately thick. But more discrepancy seen for transverse normal stresses also results from the different methods used to obtain the stresses. The present results for σ_z are determined by substituting the interlaminar shear stresses σ_{yz} and σ_{xz} achieved from the constitutive relations into the equilibrium equations of three-dimensional elasticity. Accordingly, through-thickness variation of σ_z in each lam-

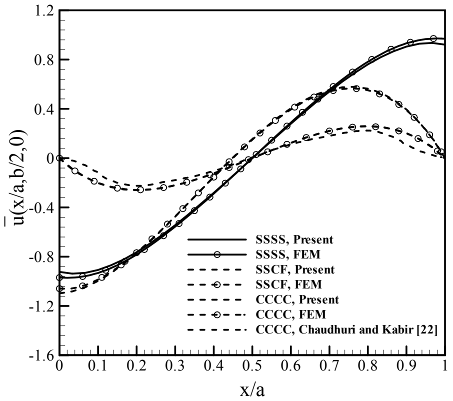
Table 3. Results of the present method and the Levy method for $[0°/90°]_s$ square plate under a double-sinusoidal pressure (a/h=50).

Method		SSSS	CSCS	FSFS	CSFS	CSSS	FSSS
Present method	$\bar{w}(a/2,a/2)$	0.4337	0.1159	1.9841	0.6062	0.2071	1.3964
Levy's method		0.4337	0.1159	1.9841	0.6062	0.2071	1.3964
Present method	$\bar{\sigma}_x(a/2,a/2,h/2)$	0.5382	0.2361	0.1330	0.0117	0.3228	0.2862
Levy's method		0.5382	0.2361	0.1330	0.0117	0.3228	0.2862
Present method	$\bar{\sigma}_y(a/2,a/2,h/4)$	0.2705	0.0727	1.2261	0.3745	0.1294	0.8639
Levy's method		0.2705	0.0727	1.2261	0.3745	0.1294	0.8639
Present method	$\bar{\sigma}_{xy}(a/4,a/4,h/4)$	-0.0053	-0.0019	-0.0010	-0.0077	-0.0032	0.0105
Levy's method		-0.0053	-0.0019	-0.0010	-0.0077	-0.0032	0.0105
Present method	$\bar{\sigma}_{yz}(a/4,a/4,0)$	0.0693	0.0137	0.3777	0.0288	0.0189	0.3716
Levy's method		0.0694	0.0137	0.3777	0.0288	0.0189	0.3716
Present method	$\bar{\sigma}_{xz}(a/4,a/4,0)$	0.1695	0.1962	0.0577	0.3090	0.2679	0.0861
Levy's method		0.1695	0.1962	0.0577	0.3090	0.2679	0.0861
Present method	$\bar{\sigma}_z(a/4,a/4,0)$	-0.2500	-0.2501	-0.2500	-0.2500	-0.2500	-0.2500
Levy's method		-0.2500	-0.2500	-0.2500	-0.2500	-0.2500	-0.2500

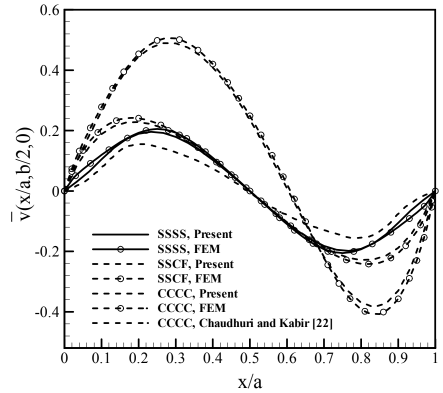
Table 4. Non-dimensionalized displacement and stress components versus n^* for $[0°/90°]_s$ square plate under a uniform pressure (a/h=50).

	n=1	n=2	n=3	n=4	n=5	Umasree and Bhaskar [13]
$\bar{u}(a/4,a/2,h/2)$	-0.2101	-0.2122	-0.2122	-0.2122	-0.2122	-
$\bar{v}(a/2,a/4,h/2)$	-0.2129	-0.2162	-0.2162	-0.2162	-0.2162	-
$\bar{w}(a/2,a/2)$	0.1583	0.1590	0.1590	0.1590	0.1590	0.1590
$\bar{\sigma}_x(a/2,a/2,h/2)$	0.2785	0.2842	0.2847	0.2847	0.2848	0.2848
$\bar{\sigma}_y(a/2,a/2,h/4)$	0.1083	0.1110	0.1112	0.1112	0.1113	0.1113
$\bar{\sigma}_{xy}(a/4,a/4,h/4)$	-0.0030	-0.0031	-0.0031	-0.0031	-0.0031	-0.0031
$\bar{\sigma}_{yz}(a/4,a/4,0)$	0.0441	0.0382	0.0377	0.0377	0.0377	-
$\bar{\sigma}_{xz}(a/4,a/4,0)$	0.2116	0.2040	0.2040	0.2040	0.2042	-
$\bar{\sigma}_z(a/4,a/4,0)$	0.4808	0.4872	0.4980	0.4978	0.4998	-

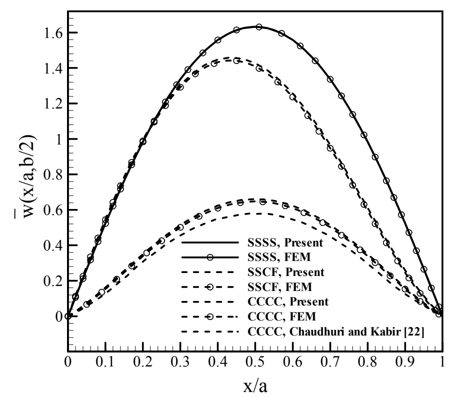
* The total number of summed terms in the assumed separated displacement field



(a)

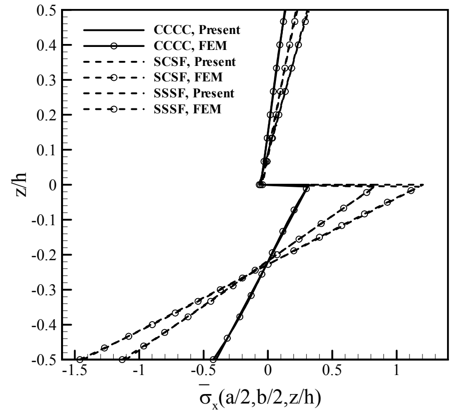


(b)

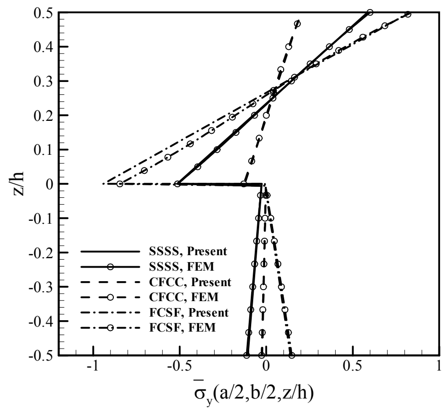


(c)

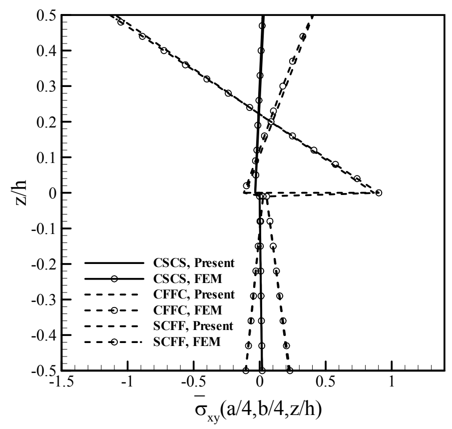
Fig. 5. Variations of the (a) axial, (b) lateral, and (c) transverse displacement vs. x at $y=b/2$ for [0°/60°] laminate compared to the results of FEM and Chaudhuri and Kabir [22].



(a)



(b)

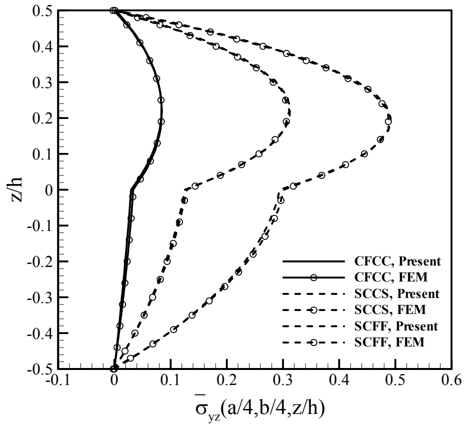


(c)

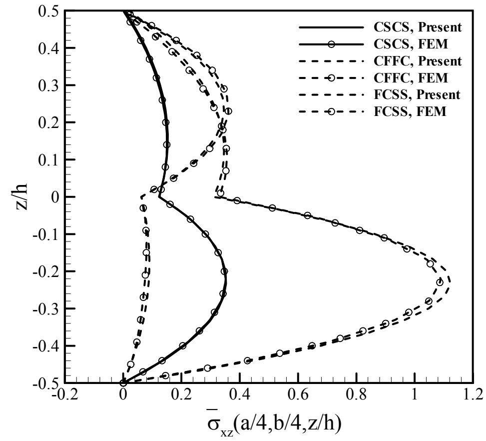
Fig. 6. Distributions of the in-plane stresses through-thickness of [0°/60°] laminate compared to the results of FEM for various sets of boundary conditions: (a) σ_x, (b) σ_y, and (c) σ_{xy}.

ina is obtained linearly.

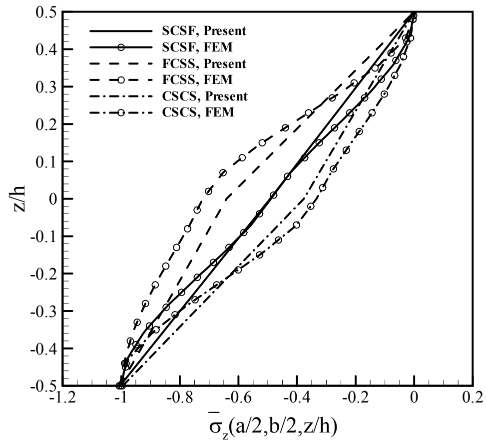
In addition, the present results of displacement components for the case of fully clamped laminated plate are compared with those given by Chaudhuri and Kabir [22] in Fig. 5. It is observed that the distributions of displacement components given by Chaudhuri and Kabir [22] are close to those obtained from the present method and FEM; nevertheless, their



(a)



(b)



(c)

Fig. 7. Distributions of the interlaminar stresses through-thickness of [0°/60°] laminate compared to the results of FEM for various sets of boundary conditions: (a) σ_{yz}, (b) σ_{xz}, and (c) σ_z.

results do not possess a sufficient accuracy.

5. Conclusions

A semi-analytical solution based on idea of the EKM has been developed to study the bending behavior of laminated composite plates. Utilizing the multi-term version of the EKM enables us to accurately analyze laminated plates with arbitrary lamination and boundary conditions. Also, the procedure used is simple and straightforward and can, therefore, be adopted in developing higher-order shear deformation and layerwise laminated plate theories. The Levy-type solutions for general cross-ply and antisymmetric angle-ply laminated plates, based on FSDT are used as a benchmark. Several numerical examples, including laminated plates with cross-ply, antisymmetric angle-ply, and general laminations with various sets of boundary condition, are studied. The numerical results are compared with those of the Levy-type solutions and also with those of the published results and finite element analysis when there exist no Levy-type solutions. All the numerical results demonstrate the capability of the proposed method for the analysis of laminated plates with arbitrary lamination and boundary conditions as well as its excellent accuracy. A convergence study has been performed to investigate the effect of number of summed terms in the assumed separated displacement field (n) on the preciseness of the numerical results obtained from the present method. It is generally found that increasing n improves the accuracy of the results, and usually by using five terms a desired accuracy for the displacements and stresses is obtained.

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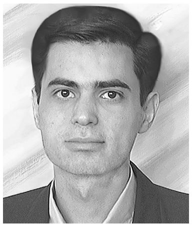

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