7^{th} Seminar on Probability and Stochastic Processes
Isfahan University of Technology
Aug. 13-14, 2009

Characterizations of Independence Under Certain Negative Dependence Structures

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Abstract

In this paper, we provide counterexamples to show that some concepts of negative dependence are strictly stronger than others. In addition, we solve an open problem posed by Hu,et.al.(2005) referring to whether strong negative orthant dependence implies that negative superadditive dependence. A characterization of independence via moment conditions is shown to hold in the class of negative upper orthant dependence random variables. Moreover, if $(X_1, X_2, ..., X_n)$ is a correlated normal random vector, we construct independence random variables $Y_1, Y_2, ..., Y_n$ and Matrix $\mathbf{A}$, such that $\mathbf{Y} = \mathbf{AX}$ under certain negative dependence structures.

MSC 2000: Primary 62G10,62G05, 62H20; Secondary 62E15, 62E20.
keywords: Multivariate negative dependence, Characterization of independence, Uncorrelated, Independence.

1 Introduction and Preliminaries

Various results in probability and statistics have been derived under the assumption that some underlying random variables have the negative dependence property. Several concepts of negative dependence have been introduced in recent years. Some of them can be derived from positive dependence ordering by compairing a random vector with a vector having the same marginals, but independent components. For instance negative superadditive and negative orthant dependence are of this type (Shaked and Shanthikumar, 2007).Many implications among different dependence concepts are well known. The reader is referred to Joe(1997), Hu (2000), Hu and Yang (2004) and Hu et.al.(2004, 2005) for an extensive treatment of the topic. Furthermore, the characterization of stochastic independence via uncorrelatedness has been studied by many authors in some classes of negative or positive dependence. For example, Lehmann (1966) proved such a characterization for the positive and negative quadrant dependence random variables, R*ü*schendorf (1981) characterized the stochastic independence in the class of upper positive orthant dependence under some suitable moment conditions. Hu, (2000) proved that if $X_1, ..., X_n$ are negatively superadditive dependent and uncorrelated random variables then $X_1, ..., X_n$ are stochastic independent. Block and Fang (1988, 1990) characterized the stochastic independence for some dependence structures also Joag-Dev (1983) characterized the stochastic independence in classes of negative association and strong negative orthant dependence random variables via uncorrelatedness. This paper is organized as follows: Section 1 recalls some well known concepts of negative dependence and presents some well known implications from them. In section 2, we provide several counterexamples and show that some concepts of negative dependence are strictly stronger than others. Moreover, we solve an open problem posed by Hu,et.al.(2005) referring to whether strong negative orthant dependence implies that negative superadditive dependence. Moreover, we characterize stochastic independence in the class of upper negative orthant dependence random variables. We, Construct independence random variables via uncorrelatedness, under certain negative dependence structures in Section 4.

Definition 1: A function $\phi : R^m \to R$ is called superadditive if for all $x, y \in R^m$

$$\phi(x \vee y) + \phi(x \wedge y) \geq \phi(x) + \phi(y),$$

where $\vee$ is for componentwise maximum and $\wedge$ is for componentwise minimum. A function $\phi(x_1, x_2, ..., x_m)$ is superadditive if and only if $\phi(x_i, x_j; x^{(i,j)})$ is superadditive in (x_i, x_j) for any $i < j$ with the other variables held fixed. This follows from Kemperman (1977). If ϕ has continuous second partial derivatives, then the superadditivity of ϕ is equivalent to $\frac{\partial^2 \phi}{\partial x_i \partial x_j} \geq 0, 1 \leq i \neq j \leq m$. (Müller and Scarsini, 2000).

Let $(X_1, ..., X_n), n \geq 3$ be a random vector defined on a probability space $(\Omega, \mathcal{F}, P)$.

Definitions 2: A random vector $(X_1, ..., X_n)$ is said to be:

(a) (Joag-Dev and Proschan, 1983). Negatively associated (NA) if for every pair of disjoint nonempty subsets A_1, A_2 of $\{1, 2, ..., n\}$,

$$Cov(f_1(X_i, i \in A_1), f_2(X_j, j \in A_2)) \leq 0,$$

whenever f_1 and f_2 are coordinatewise nondecreasing functions and covariance exists.

(b) Weakly negatively associated (WNA) if for all nonnegative and nondecreasing functions $f_i, i = 1, 2, ..., n$

$$E(\Pi_{i=1}^n f(X_i)) \leq \Pi_{i=1}^n Ef(X_i)$$

(c) Negatively upper orthant dependent (NUOD) if for all $x_i \in R, i = 1, 2, ..., n$

$$P(\bigcap_{i=1}^{n} [X_i > x_i]) \leq \prod_{i=1}^{n} P(X_i > x_i), \tag{1}$$

Negatively lower orthant dependent (NLOD) if for all $x_i \in R, i = 1, 2, ..., n$

$$P(\bigcap_{i=1}^{n} [X_i \leq x_i]) \leq \prod_{i=1}^{n} P(X_i \leq x_i), \tag{2}$$

And negatively orthant dependent (NOD), if both (1) and (2) hold.

(d) (Hu,2000). Negatively superadditive dependent (NSD) if

$$E\phi(X_1, X_2, ..., X_m) \leq E\phi(Y_1, Y_2, ..., Y_m), \tag{3}$$

where $Y_1, Y_2, ..., Y_m$ are independent with $X_i =^{st} Y_i$ for each i and ϕ is a superadditive function such that the expectations in (3) exist.The concepts of positively superadditive dependent is defined with reversing inequality in (3).

(e) Linearly negative dependent (LIND) if for any disjoint subsets A and B of $\{1, 2, ..., n\}$ and $\lambda_i \geq 0, i = 1, 2, ..., n$, the random variables $\sum_{i \in A} \lambda_i X_i$ and $\sum_{j \in B} \lambda_j X_j$ are negative quadrant dependent.

(f) (Joag-Dev, 1983). Strongly negative orthant dependent (SNOD) if for every set of indices A in $\{1, 2, ..., n\}$ and for all $x_i \in R, i = 1, 2, ..., n$, the following three conditions hold,

- $P[\bigcap_{i=1}^n (X_i > x_i)] \leq P[X_i > x_i, i \in A] P[X_j > x_j, j \in A^c]$,
- $P[\bigcap_{i=1}^n (X_i \leq x_i)] \leq P[X_i \leq x_i, i \in A] P[X_j \leq x_j, j \in A^c]$,

- $P[X_i > x_i, i \in A, X_j \leq x_j, j \in A^c] \leq P(X_i > x_i, i \in A)P(X_j \leq x_j, j \in A^c)$.

The following implications are well known.
i) If $(X_1, ..., X_n)$ is NA then it is LIND, WNA and consequence NUOD.
ii) If $(X_1, ..., X_n)$ is NA then it is NSD.(Christofides and Vaggelatou, 2004).
iii) If $(X_1, ..., X_n)$ is NSD then it is NUOD.(Hu, 2000).
iv) If $(X_1, ..., X_n)$ is NA then it is SNOD and if $(X_1, ..., X_n)$ is SNOD then it is NOD.(Joag-Dev, 1983)

It is well known that some of negative dependence concepts do not imply others.
i) Neither of the two dependence concepts NUOD and NLOD implies the other (Bozorgnia et.al, 1996)
ii) Neither NUOD nor NLOD imply NA. (Joag-Dev and Proschan, 1983).
iii) The NSD does not imply LIND and NA.(Hu, 2000).
iv) The NSD does not imply SNOD. (Hu, et.al., 2005).

2 Some counterexamples

In this section, we present some counterexamples showing that some concepts of negative dependence are strictly stronger than others. Throughout this section, $p(i, j, k)$ will denote $P[X_1 = i, X_2 = j, X_3 = k]$.
Lemma 1: Neither of the two dependence concepts SNOD and LIND implies the other.
Proof: i) (LIND does not imply SNOD). Let (X_1, X_2, X_3) have the following joint probability function.

$$p(1,1,1) = 0.05, p(1,0,0) = p(0,1,0) = 0.225, p(0,0,1) = 0.22,$$

$$p(0,0,0) = 0.065, p(1,1,0) = 0.08, p(0,1,1) = 0.06, p(1,0,1) = 0.075,$$

It can be shown that (X_1, X_2, X_3) is LIND and NOD. However,

$$P[\bigcap_{i=1}^{3}(X_i > \frac{1}{2})] = 0.05 > P[X_1 > \frac{1}{2}]P[X_2 > \frac{1}{2}, X_3 > \frac{1}{2}] = 0.0473,$$

establishing that (X_1, X_2, X_3) is not SNOD.
ii) (SNOD does not imply LIND). Let (X_1, X_2, X_3, X_4) have the joint probability function as given in Table 6 of Hu, et.al.(2005). Then (X_1, X_2, X_3, X_4) is SNOD but not LIND, since

$$P[Y_1 \geq 1, Y_2 \geq 2] = \frac{9}{32} > P[Y_1 \geq 1].P[Y_2 \geq 2] = \frac{8}{32},$$

Where $Y_1 = X_1, Y_2 = X_2 + X_3 + X_4$.
The next Lemma indicates that strong negative orthant dependence does not imply NSD which gives the answer to the question posed by Hu, et.al.(2005).
Lemma 2: The SNOD does not imply NSD.
Proof: Consider a random vector (X_1, X_2, X_3) with the following joint probability function.

$$p(1,1,1) = p(0,0,2) = p(0,2,0) = p(2,0,0) = \frac{1}{40}, p(1,0,0) = \frac{2}{40}.$$

$$p(0,0,1) = p(0,1,0) = \frac{2}{40}, p(1,1,0) = p(0,1,1) = p(1,0,1) = \frac{10}{40}.$$

It can be verified that (X_1, X_2, X_3) is SNOD, since for all $a_i, b_i \in R$, $i = 1, 2, 3$. The conditions of Definition 2 (f) are hold. But (X_1, X_2, X_3) is not NSD. Let $f(x_1, x_2, x_3) = \max\{x_1 + x_2 + x_3 - 1, 0\}$, this function is supermodular since it is a composition of an increasing convex real value function and an increasing supermodular function. For this function we get

$$Ef(X_1, X_2, X_3) = \frac{56000}{64000} > Ef(Y_1, Y_2, Y_3) = \frac{50494}{64000}.$$

Where Y_1, Y_2, Y_3 are independent random variables with $X_i =^{st} Y_i$ for each $i = 1, 2, 3$.
The following example illustrates that the converse implication $NA \Rightarrow LIND$ fails to hold.
Example 2 : Let (X_1, X_2, X_3) have the following joint probability function.

$$p(0,0,0) = 0, p(0,0,1) = p(1,0,0) = \frac{2}{15}, p(1,1,1) = \frac{3}{15},$$

$$p(0,1,0) = p(1,1,0) = p(0,1,1) = p(1,0,1) = \frac{2}{15}.$$

It is easy to show that (X_1, X_2, X_3) is LIND. However, for the following increasing functions,

$$f(x_1, x_2) = \begin{cases} (x_1 - \frac{1}{15})(x_2 - \frac{1}{15}), & \text{if } x_1 > 0.5,\ x_2 > 0.5 \\ \frac{1}{15^2}, & \text{if } x_1 \le 0.5,\ x_2 \le 0.5 \end{cases}$$

and

$$g(x_3) = \begin{cases} (x_3 - \frac{1}{15}), & \text{if } x_3 > 0.5 \\ -\frac{1}{15}, & \text{if } x_3 \le 0.5 \end{cases}$$

we have $Cov(f(X_1, X_2), g(X_3)) > 0$. So, (X_1, X_2, X_3) is not NA.
Example 3: (NOD implies neither NA nor LIND). Let (X_1, X_2, X_3) have the following joint probability function:

$$p(0,0,0) = p(1,0,1) = 0, p(0,1,0) = p(0,0,1) = \frac{2}{10}$$

$$p(0,1,1) = p(1,1,0) = p(1,1,1) = \frac{1}{10}, p(1,0,0) = \frac{3}{10}.$$

- It is easy to see that (X_1, X_2, X_3) is ND ,
- If $f(x_1, x_2) = I_{(x_1 > \frac{1}{2}, x_2 > \frac{1}{2})}$ and $g(x_3) = I(x_3 > \frac{1}{2})$, then $E(f(X_1, X_2).g(X_3)) > Ef(X_1, X_2).Eg(X_3)$. Therefore, (X_1, X_2, X_3) is not NA.
- The random vector (X_1, X_2, X_3) is not LIND. Since, the random variables $Y_1 = X_1 + X_2$ and $Y_2 = X_3$ are PQD.
- The NOD does not imply SNOD, because for $0 < a_i < 1, i = 1, 2, 3$, we have

$$P[\bigcap_{i=1}^{3}(X_i > a_i)] = \frac{1}{10} > P[X_1 > a_1, X_2 > a_2].P[X_3 > a_3] = \frac{4}{50}.$$

2.1 Characterization of independence

It is well known that uncorrelatedness of two random variables X and Y does not imply stochastic independence of X and Y, except in special cases. It is important to charactrize stochastic independence via uncorrelatedness under certain dependence structures. Several authors have discussed dependence structures in which uncorrelatedness implies independence. Among them are Lehmann (1966), Jogdeo (1983), Block and Fang (1988, 1990) and Hu (2000). Now, we characterize the stochastic independence in the class of NUOD random variables under condition $E\prod_{i\in T} X_i = \prod_{i\in T} EX_i$, for all $T \subset \{1, 2, ..., n\}$. Let $\mathbf{X} = (X_1, X_2, ..., X_n)$ be a random vector, and $\mathbf{X}^{\perp}$ be another n-dimensional random vector whose univariate marginal distributions coincide with the marginals of $\mathbf{X}$, and whose components are independent. Then, the concept of NUOD is equivalent to $\mathbf{X}$ is smaller than $\mathbf{X}^{\perp}$ in the upper orthant order (denoted by $\mathbf{X} \le_{uo} \mathbf{X}^{\perp}$). So, Theorem 6.G.1 in Shaked and Shanthikumar, (2007) implies that, $\mathbf{X} \le_{uo} \mathbf{X}^{\perp}$ if and only if,

$$E\prod_{i=1}^{n} f_i(X_i) \le \prod_{i=1}^{n} Ef_i(X_i), \tag{4}$$

for every collection $\{f_1, f_2, ..., f_n\}$ of univariate nonnegative increasing functions. Therefore, NUOD is equivalent to WNA. Also, recall that, Lehmann (1966) proved that NUOD of X_1 and X_2 is equivalent to $Cov(f(X_1), g(X_2)) \le 0$, for all nonnegative and nondecreasing Borel functions f and g . Therefore

NUOD is equivalent to weak negative association for $n = 2$.
Remark 1: The condition of non-negativity functions $f_i, i = 1, 2, ..., n$ in (4) is a necessary condition. To see this consider Example 3. If $f_1(x) = f_2(x) = I_{(x>\frac{1}{2})}$ and $f_3(x) = x - \frac{1}{4}$. Then

$$E\{f_1(X_1)f_2(X_2)f_3(X_3)\} > Ef_1(X_1)Ef_2(X_2)Ef_3(X_3).$$

Corollary 1: Let $(X_1, X_2, ..., X_n)$ be a non-negative NUOD random vector, then, by a simple generalization of Theorem 1 of Rüschendorf (1981), $E(\prod_{i=1}^n X_i) = \prod_{i=1}^n E(X_i)$ implies independence of $(X_1, X_2, ..., X_n)$.
Corollary 2: Let $(X_1, X_2, ..., X_n)$ be NUOD a random vector assuming that $E\prod_{i\in T} X_i$, exists for all $T \subset \{1, 2, ..., n\}$. Then, Theorem 2 in Block and Fang (1988) implies that, $E\prod_{i\in T} X_i = \prod_{i\in T} EX_i$, for all $T \subset \{1, 2, ..., n\}$, if and only if $X_1, X_2, ..., X_n$ are independent.

3 Construction Independence random variables

Let $(X_1, X_2, ..., X_n)$ be an arbitrary random vector, then in many statistical problems, it is useful to construct uncorrelated random variables $Y_1, Y_2, ..., Y_n$ and real valued function
$f : R^n \mapsto R$, such that $\mathbf{X} = f(\mathbf{Y})$. In particular, f can be a linear function. It seems to be important to find real valued functions $g_1, g_2, ..., g_n$ that make $g_1(Y_1), g_2(Y_2), ..., g_n(Y_n)$ uncorrelated. Gupta, et.al.(2000) have been proved the existence of real valued functions $f_1, f_2, ..., f_n$ such that $(X_1, X_2, ..., X_n) = (f_1(Y_1), ..., f_n(Y_n))$, *a.e.* where $Y_1, Y_2, ..., Y_n$ are uncorrelated.
The following theorem explain the approach of Gupta et.al.(2000) in constructing uncorrelated random variables.
Theorem 1 Let $(X_1, X_2, ..., X_n)$ be an arbitrary random vector with covariance matrix
$\Sigma = (\sigma_{ij})$. Then, there exist indicator random variables $U_1, U_2, ..., U_n$ such that $(U_1, U_2, ..., U_n)$ is independence of $(X_1, X_2, ..., X_n)$ and $Y_i = X_i + c.U_i$, *a.e.* $i = 1, 2, ..., n$ are uncorrelated (if c is suitably chosen) and Y_i uniquely determines X_i (and U_i).
Remark 2 *i*) Without loss of generality, we may suppose that $0 \le X_i \le 1, i = 1, 2, ..., n$, otherwise apply the one-to-one transforms $Y_i = \frac{1}{2} + \frac{1}{2}\arctan(X_i)$, *a.e.*$i = 1, 2, ..., n$.
ii) We can easily reconstruct X_i (and U_i) from Y_i, this is where we use the restriction $0 \le X_i \le 1$, then $X_i = Y_i - c[\frac{Y_i}{c}]$, *a.e.* where [.] denotes the integer part.(Gupta, et.al.2000).
Lemma 3 Let $(X_1, X_2, ..., X_n)$ be a NA random vector which is independent of $(U_1, U_2, ..., U_n)$. Then for any constant c, the random variables $Y_i = X_i + c.U_i, a.e.$ $i = 1, 2, ..., n$ are NA.
Proof Let f and g be real valued and coordinatewise nondecreasing functions, then for every pair of disjoint nonempty subsets A and B of $\{1, 2, ..., n\}$, we have

$$\begin{aligned} E(f(Y_i, i \in A)g(Y_j, j \in B)) &= E\{E[f(X_i + cU_i, i \in A)g(X_j + cU_j, j \in B)|U_1, ..., U_n]\} \\ &\le E\{E[f(X_i + cU_i, i \in A)|U_1, ..., U_n]E[g(X_j + cU_j, j \in B)|U_1, ..., U_n]\} \\ &= E\{f(Y_i, i \in A)\}E\{g(Y_j, j \in B)\} \end{aligned}$$

Remark 3 Under the assumptions of Theorem 1. Let the random vector $(X_1, X_2, ..., X_n)$ be pairwise negative dependent (PND), then by Theorem 1 in Lehmann (1966) the random variables $Y_i = X_i + c.U_i$, *a.e.* $i = 1, 2, ..., n$ are PND. Also, if the random vector $(X_1, X_2, ..., X_n)$ is NSD, then Corollary 9.A.10 in Shaked and Shanthikumar (2007) implies that $Y_i = X_i + c.U_i, a.e.$ $i = 1, 2, ..., n$ are NSD. Therefore, the following results are hold in classes of NA, NSD and pairwise negative dependent of random variables.
Now, we apply the approach of Gupta,et.al.(2000) for construction independence random variables for some negative dependence structures.
Corollary 3 Under the assumptions of Theorem 1, let the random vector $(X_1, X_2, ..., X_n)$ be negative association, Then

- If $(N_1, N_2..., N_n)$ is a normal random vector with correlation matrix $R = (\rho_{ij})$ where $(N_1, N_2..., N_n)$ is independent of $(X_1, X_2, ..., X_n)$ and $\rho_{ij} = -\sin(\frac{\pi}{2c^2}\sigma_{ij})$. We define $U_i = sgn(N_i), i = 1, 2, ..., n$, and selecting a constant c such that $R = (\rho_{ij})$ become positive definite, then for all $i \ne j$, we have $Cov(Y_i, Y_j) = \sigma_{ij} + c^2.Cov(U_i, U_j) = 0$ where $Cov(U_i, U_j) = 4P(N_i \le 0, N_j \le 0) - 1 = \frac{2}{\pi}\arcsin(\rho_{ij})$

for all $i \neq j$. Therefore, by Lemma 3 $(Y_1, Y_2, ..., Y_n)$ is NA and by Theorem 1 in Joag-Dev (1983), $(Y_1, Y_2, ..., Y_n)$ is independent.

- Let $(N_1, N_2 ..., N_n)$ be a random vector with n-dimensional FGM distribution. Then, the marginal bivariate distribution functions are given by,

$$F_{ij}(x_i, y_j) = F_i(x_i)F_j(y_j)[1 + \alpha_{ij}(1 - F_i(x_i))(1 - F_j(x_j))], \quad 1 \leq i < j \leq n.$$

Where $F_i(x_i)$ is a specified cumulative distribution function, $i = 1, 2, ..., n$ and for all $1 \leq i < j \leq n$, $|\alpha_{ij}| < 1$ (Mari and Kotz, 2001).
i)-If the margin of distributions are $N(0, \sigma_i^2)$, then

$$Cov(Y_i, Y_j) = \sigma_{ij} + c^2 Cov(U_i, U_j) = 0 \Leftrightarrow \alpha_{ij} = 12 - \frac{16\sigma_{ij}}{c^2}.$$

Which, selecting the constant c such that $|\alpha_{ij}| < 1$, implies independence of $(Y_1, Y_2, ..., Y_n)$.
ii)- If the margin of distributions are Exponential with parameters $\lambda_i, i = 1, 2, ..., n$, then $P(N_i \leq 0, N_j \leq 0) = 0$, so $Cov(U_i, U_j) = -1$ for all $1 \leq i < j \leq n$. This implies that

$$Cov(Y_i, Y_j) = 0 \Leftrightarrow \sigma_{ij} = c^2.$$

Remark 4 In Corollary 3, we assumed that the random vector $(X_1, X_2, ..., X_n)$ is NA then $\sigma_{ij} \leq 0$ for all $i \neq j$. However, we observe that in FGM family distributions with exponential marginal distributions $Cov(Y_i, Y_j) = 0 \Leftrightarrow \sigma_{ij} = c^2, \quad 1 \leq i < j \leq n$, that is a construction to approach of Gupta et.al.(2000) which is not general.

3.1 Construction of independence normal variables

It is well known that for a normally distributed n-dimensional random vector stochastically independence is equivalent to $Cov(X) = I$ -the identity matrix. Moreover, if $(X_1, X_2, ..., X_n)$ is a normal random vector with Covariance matrix $\Sigma = (\sigma_{ij})$, then the following results are well known:

- The random vector $(X_1, X_2, ..., X_n)$ is NA if $\sigma_{ij} \leq 0$ for all $i \neq j, \ i, j = 1, 2, ..., n$. (Joag-Dev and Proschan, 1983).
- The random vector $(X_1, X_2, ..., X_n)$ is NSD if $\sigma_{ij} \leq 0$ for all $i \neq j, \ i, j = 1, 2, ..., n$. (Hu, 2000).

In the following, we will show that, how to construct the linear function **Y=AX**, (where $|A| \neq 0$) from correlated normal random variables, that $(Y_1, Y_2, ..., Y_n)$ is independent.
Example 4 Let (X_1, X_2) have a $N_2(\mathbf{0}, \Sigma)$ distribution with $\Sigma = (\sigma_{ij})$ and $\sigma_{12} \neq 0$. Set,

$$Y_1 = a_{11}X_1 + a_{12}X_2, Y_2 = a_{21}X_1 - a_{22}X_2$$

such that $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Then it is easy to verify that:
i) The random vector (Y_1, Y_2) have a $N_2(\mathbf{0}, \Sigma_y)$ distribution with

$$Cov(Y_1, Y_2) = a_{11}a_{21}\sigma_{11} + a_{12}a_{22}\sigma_{22} + (a_{11}a_{22} + a_{12}a_{21})\sigma_{12}.$$

So, (Y_1, Y_2) is NA if $Cov(Y_1, Y_2) \leq 0$.
ii) In particular, for $\sigma_1 = \sigma_2 = \sigma$ if $\rho_{12} = -\frac{a_{11}a_{21}+a_{12}a_{22}}{a_{11}a_{22}+a_{12}a_{21}}$ with $a_{11}a_{22} + a_{12}a_{21} \neq 0$ then Theorem 1 in Joag-Dev (1983) yield independence of (Y_1, Y_2). In this case $\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y}$ where $\mathbf{A} = (a_{ij})$.
Example 5 Let (X_1, X_2, X_3) have a $N_3(\mathbf{0}, \Sigma)$ distribution with $\Sigma = (\sigma_{ij})$ and $\sigma_{ij} \neq 0$ for all $i \neq j$. We define **Y=A.X** with $|A| \neq 0$. Then it is easy to show that (Y_1, Y_2, Y_3) is $N_3(\mathbf{0}, \Sigma_y)$, therefore the random vector (Y_1, Y_2, Y_3) is NA if $Cov(Y_i, Y_j) \leq 0$ for all $i \neq j$ and is independent if $Cov(Y_i, Y_j) = 0$. So, we can construct independence random variables into dependence ones. In particular, if $A = (a_{ij})$ where $a_{ii} = -1$ and $a_{ij} = 1, (i \neq j), i, j = 1, 2, ..., n$. Then, $Corr(X_i, X_j) \leq \frac{\sigma_i+\sigma_j-\sigma_k}{2\sigma_i\sigma_j}$ for all $i \neq j$ and $k \neq i, j$ yield negative association of (Y_1, Y_2, Y_3) and independence if $Corr(X_i, X_j) = \frac{\sigma_i+\sigma_j-\sigma_k}{2\sigma_i\sigma_j}$. In

addition, if $\sigma_1 = \sigma_2 = \sigma_3$ and $-1 \leq \rho_{ij} \leq \frac{1}{2}$ for all $i \neq j$, then the random vector (Y_1, Y_2, Y_3) is NA and is independent if $\rho_{ij} = \frac{1}{2}$ for all $i \neq j$. In this case $X_1 = \frac{Y_2+Y_3}{2}, X_2 = \frac{Y_1+Y_3}{2}, X_3 = \frac{Y_1+Y_2}{2}$
Example 6 Let $(X_1, X_2, ..., X_n)$ $(n > 3)$ be a multivariate normal random vector with $\sigma_i = \sigma, i = 1, 2, ..., n$ and $Corr(X_i, X_j) = \rho \leq 0$ for all $i \neq j$. If $A = (a_{ij})$, where $a_{ii} = -1$ and $a_{ij} = 1, (i \neq j), i, j = 1, 2, ..., n$. Then **Y=A.X** is a NA normal random vector if $-1 \leq \rho \leq -\frac{1}{n-3}$. Also $(Y_1, Y_2, ..., Y_n)$ is independent if $\rho = -\frac{1}{n-3}$.

4 Conclusions

- The counterexamples have been presented in Sections 1 and 2 show that the following implications holding among these concepts of dependence are strict for all $n \geq 3$:

$$\begin{array}{c} NUOD \Leftarrow NOD \Leftarrow SNOD \Leftarrow \quad NA \quad \Rightarrow NSD \Rightarrow NUOD \Leftrightarrow WNA \\ \Downarrow \\ LIND \end{array}$$

- The characterization of stochastic independence in smaller class LIND , is still an open problem.
- In approach of Gupta et.al.(2000), we cannot hope that there always exist one-to-one functions f_i, with which Theorem 1 holds. Moreover, we can construct the linear transforms of correlated normal random variables which are independent under some dependence structures.

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