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Asymptotic Behavior of Product of Two Heavy-Tailed Dependent Random Variables

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Abstract

Let X and Y be positive weakly negatively dependent (WND) random variables with finite expectations and continuous distribution functions F and G with heavy tails, respectively. The asymptotic behavior of the tail of distribution of XY is studied and some closure properties under some suitable conditions on $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$ are provided. Moreover, subexponentiality of XY when X and Y are WND random variables is derived.

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1 Introduction

The subexponentiality of distribution of XY when X and Y are independent heavy tailed random variables with distribution functions F and G respectively, has been studied by Cline and Samorodnitsky[3]. They proved that, if F belongs to the class of subexponential distributions (S), denoted by $F \in S$ under some suitable conditions on $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$, then the distribution of XY belongs to S. Tang [9] by removing conditions of Cline and Samorodnitsky and adding a mild condition to the distribution F , extended these results. Following the works of Tang [9] and Cline and Samorodnitsky[3], we will study asymptotic behavior of the tail distribution of XY, when X and Y are WND random variables with finite expectations and continuous distribution functions F and G, respectively. In fact, we prove, if F and G belong to classes L or D, under some suitable conditions on $\bar{F}$ and $\bar{G}$, then the distribution of XY also belongs to the L or D. Finally, we derive subexponentiality of XY when X and Y are WND random variables.

An important class of heavy tailed distributions is D, which consists of all distributions with dominated variation. By definition, a distribution function F belongs to the class D, if $\limsup \bar{F}(xy)/\bar{F}(x) < \infty$, holds for some $0 < y < 1$ as $x \to \infty$. A wider class of heavy tailed distributions is L, which consists of all distributions with long tailed distributions. By definition, a distribution function belongs to L, if $\lim \bar{F}(x-y)/\bar{F}(x) = 1$ holds for any $y \in R$, as $x \to \infty$. For a distribution F with $\bar{F}(x) > 0$ for all $x \geq 0$, the *lower Matuszewska index* of the function F is defined as follow,

$$J_*(F) = \sup\left\{-\frac{\bar{F}^*(\upsilon)}{\log \upsilon}\right\} \quad with \quad \bar{F}^*(\upsilon) = \limsup_{x\to\infty}\frac{\bar{F}(\upsilon x)}{\bar{F}(x)} \;\; for \;\; \upsilon > 1.$$

It is easy to see that, the condition $0 < J_*(F) \le \infty$ is equivalent to condition $\bar{F}^*(\upsilon) < 1$, for some $\upsilon > 1$. For details of the *lower Matuszewska indices* see Bingham et al[1]. (chapter 2.1) and for further discussions and applications see Cline and Samorodnitsky [3]and Tang and Tistsiashvili [10]. Throughout this paper all distribution functions are defined on $[0,\infty)$ and $f(x) \sim g(x)$ means that $\lim f(x)/g(x) = 1$ as $x \to \infty$. We denote the tail of distribution of F by $\bar{F}(x) = 1 - F(x)$ and distribution of product of XY by H, say $H(t) = P(XY \le t)$. The Weakly Negative Dependence (WND), which is introduced as follows, is a kind of dependence which has some good and simple property that allows us to prove some useful results.

Definition 1.1. *The random variables X and Y are said Weakly Negatively Dependent (WND) if there exists some $C > 1$ such that, $f(x,y) \le C.f_1(x).f_2(y)$ where $f(x,y)$, $f_1(x)$ and $f_2(y)$ are joint density and marginal densities of X and Y, respectively.*

Remark 1.2. *Let X and Y be two WND random variables with distribution functions F and G respectively. Then it is easy to show that,*

i) For every $x, y \in R$ we have,$F_{X,Y}(x,y) \le C.F(x)G(y)$.
ii) For all positive value of x, $P(X+Y > x) \le C.\int_0^\infty \bar{F}(x-u)dG(u)$.
iii) If $h_1(.)$ and $h_2(.)$ are monotone measurable functions then $h_1(X)$ and $h_2(Y)$ are WND. In particular, it is valid when $h_i(x) = c_i x, i = 1,2$ where $c_i \in R$.

2 Main results

In this section, we study the asymptotic behaviors and some closure properties of classes D and L for product of two random variables with heavy tail distribution functions.

Theorem 2.1. *Let X and Y be two WND random variables with distribution functions F and G, respectively. Suppose that F and G belong to D and $0 < J_*(G) < \infty$. If there exists some $0 < p < J_*(G)$ for which $E(X^{-p}) < \infty$, then $H \in D$, where $H(x) = P(XY > x)$.*

Proof. Since we study the asymptotic behavior of tail of the distribution functions for sufficiently large positive value of x, hence, without loss of generality we assume that $x > 1$. We prove the theorem in three parts

i. If $X > 1$ $a.s.$ and $Y > 1$ $a.s.$, then we have

$$\begin{aligned}\bar{H}(x) &= P(XY > x) = P(\ln X + \ln Y > \ln x)\\ &= P(\ln X + \ln Y > \ln x; \ln X < \ln x; \ln Y < \ln x) + P(\ln X > \ln x)\\ &\quad + P(\ln Y > \ln x) - P(\ln X > \ln x; \ln Y > \ln x).\end{aligned} \tag{2.1}$$

On the other hand, it is easy to see that for any positive values a, b, c and d,

$$\frac{a+b}{c+d} \le \max\left\{\frac{a}{c}, \frac{b}{d}\right\}. \tag{2.2}$$

So, for some $0 < t < 1$, we get

$$\frac{\bar{H}(tx)}{\bar{H}(x)} \le \max\{I_1, I_2\}. \tag{2.3}$$

where,

$$\begin{aligned} I_1 &= \frac{P(X>tx)-P(X>tx,Y>tx)}{P(X>x)-P(X>x,Y>x)} \le \frac{P(X>tx)}{P(X>x)-C.P(X>x)P(Y>x)} \\ &= \left[\frac{P(X>x)}{P(X>tx)} - \frac{C.P(X>x)P(Y>x)}{P(X>tx)}\right]^{-1} = [I_3+I_4]^{-1}. \end{aligned}$$

The inequality directly follows form Remark 1.2. Then, by $F \in D$, we have, $0 < \lim_{x\to\infty} I_3 < 1$ and $\lim_{x\to\infty} I_4 = 0$, therefore, $\limsup_{x\to\infty} I_1 < \infty$. Moreover,

$$\begin{aligned} I_2 &= \frac{P(Y>tx)+P(XY>tx;Y<tx;X<tx)}{P(Y>x)+P(XY>x;Y<x;X<x)} \\ &\le \frac{P(Y>tx)}{P(Y>x)} + C.\int_1^{tx} \frac{\bar{G}(tx/u)-\bar{G}(tx)}{\bar{G}(x)} dF(u) = \frac{P(Y>tx)}{P(Y>x)} + I_5. \end{aligned}$$

Where the last equality follows from remark 1.2. Now, by Tang [9] and the second statement of Proposition 2.2.1 of Bingham et al.[1], we conclude that for each $0 < p < J_*(G)$, there exist positive constants C' and x_0 such that the inequality $\bar{G}(y)/\bar{G}(x) \le C'(y/x)^p$, holds uniformly for $x_0 \le y \le x$, or equivalently, that the inequality

$$\frac{\bar{G}(tx/u)}{\bar{G}(x)} \le C'(\frac{tx/u}{x})^p = C'(t/u)^p,$$

holds uniformly for $x_0 \le \frac{tx}{u} \le x$ or for $t \le u \le \frac{tx}{x_0}$. So we have

$$\int_t^{\frac{tx}{x_0}} \frac{\bar{G}(tx/u)}{\bar{G}(x)}\, dF(u) \le C.C'.\int_t^{\frac{tx}{x_0}} (t/u)^p\, dF(u) \le M_1.t^p.E(X^{-p}) < \infty. \tag{2.4}$$

Where, $M_1 = C.C'$. Now, if $x_0 \le 1$, then, by (2.4) we can write

$$I_5 \le C.\int_t^{tx} \frac{\bar{G}(tx/u)}{\bar{G}(x)}\, dF(u) \le C.\int_t^{\frac{tx}{x_0}} \frac{\bar{G}(tx/u)}{\bar{G}(x)}\, dF(u) < \infty.$$

If $x_0 > 1$, we have

$$\begin{aligned} I_5 \le C.\int_t^{tx} \frac{\bar{G}(tx/u)}{\bar{G}(x)}\, dF(u) &= C.\int_t^{\frac{tx}{x_0}} \frac{\bar{G}(tx/u)}{\bar{G}(x)}\, dF(u) + C.\int_{\frac{tx}{x_0}}^{tx} \frac{\bar{G}(tx/u)}{\bar{G}(x)}\, dF(u) \\ &= C.\int_t^{\frac{tx}{x_0}} \frac{\bar{G}(tx/u)}{\bar{G}(x)}\, dF(u) + I_6. \end{aligned}$$

On the other hand, we have

$$\limsup_{x\to\infty} I_6 = \int_1^{x_0} \limsup_{x\to\infty} \frac{\bar{G}(k)}{\bar{G}(uk/t)}.\frac{tx}{k^2}\, f(tx/k)\, dk = 0. \tag{2.5}$$

Where $k = tx/u$ and f is density function of X. The last equality follows from $G \in D$ and the fact that $\lim_{x\to\infty} tx/k^2 = 0$. Therefore, using (2.4) and (2.5) we can conclude that $\limsup_{x\to\infty} I_5 < \infty$ and then $\limsup_{x\to\infty} I_2 < \infty$. So

$$\limsup_{x\to\infty} \frac{\bar{H}(tx)}{\bar{H}(x)} < \infty.$$

ii. If $X > 1$ *a.s.* and $0 < Y < 1$ *a.s.*, then for some $0 < t < 1$ and for all $x > 1$, Remark 1.2 and $F \in D$ imply that

$$\limsup_{x\to\infty} \frac{\bar{H}(tx)}{\bar{H}(x)} \le \limsup_{x\to\infty} \frac{P(XY>tx)}{P(X>x)} \le \limsup_{x\to\infty} C.\int_0^1 \frac{\bar{F}(tx/u)}{\bar{F}(x)} dG(u) < \infty.$$

iii. The case $0 < X < 1$ *a.s.* and $Y > 1$ *a.s.* is similar to (*ii*).

Using (2.2), and last three section we have

$$\limsup_{x\to\infty} \frac{\bar{H}(tx)}{\bar{H}(x)} \le \limsup_{x\to\infty} \max\{\frac{P(XY > tx; X > 1; Y > 1)}{P(XY > x; X > 1; Y > 1)} \\ , \frac{P(XY > tx; X < 1; Y > 1)}{P(XY > x; X < 1; Y > 1)}, \frac{P(XY > tx; X > 1; Y < 1)}{P(XY > x; X > 1; Y < 1)}\} < \infty.$$

This completes the proof. □

Theorem 2.2. *Let X and Y be two WND random variables with distribution functions F and G, respectively. Suppose that $F, G \in D$ and $E(X) < \infty$ $(E(Y) < \infty)$. If $\bar{F}(x) = O(\bar{G}(x))$ then $H \in D$.*

Proof. The approach of the proof is similar to the proof of Theorem 2.1 just we need to change the relation (2.5). In the new situation, for all $x > 0$ and $0 < t < 1; h > 0$, we have

$$\begin{aligned} & \frac{P(XY > tx; Y < tx; X < tx)}{P(Y > x)} \le C.\int_0^{tx} \frac{\bar{G}(tx/u)}{\bar{G}(x)} dF(u) \\ \le\ & \sum_{n=0}^{N_0} \int_{nh}^{(n+1)h} \frac{\bar{G}(tx/u)}{\bar{G}(x)} dF(u) \le \frac{1}{\bar{G}(x)} \sum_{n=0}^{N_0} \bar{G}\left(\frac{tx}{(n+1)h}\right) \left[\bar{F}(nh) - \bar{F}((n+1)h)\right] \\ \le\ & \frac{\bar{G}(tx/h)\bar{F}(0)}{\bar{G}(x)} + \sum_{n=1}^{N_0} \bar{F}(nh) \left[\frac{\bar{G}(tx/((n+1)h)) - \bar{G}(tx/nh)}{\bar{G}(x)}\right] = K_1 + K_2. \end{aligned} \tag{2.6}$$

Where $N_0 = [tx/h]$. Since $G \in D$, we have $\limsup_{x\to\infty} K_1 < \infty$. On the other hand, by Theorem .... we know for any $t, x > 0$ and $h > 0$ there exist some $n_0 \in N$ such that for every $n > n_0$, $nh > tx$. So we have

$$\begin{aligned} K_2 &= \sum_{n=1}^{N_0} \bar{F}(nh) \left[\frac{\bar{G}(tx/((n+1)h)) - \bar{G}(tx/nh)}{\bar{G}(x)}\right] \\ &= \sum_{n=1}^{n_0} + \sum_{n=n_0+1}^{N_0} \bar{F}(nh) \left[\frac{\bar{G}(tx/((n+1)h)) - \bar{G}(tx/nh)}{\bar{G}(x)}\right] = K_3 + K_4. \end{aligned} \tag{2.7}$$

Now we get

$$\lim_{x\to\infty} K_3 \le \sum_{n=1}^{n_0} \bar{F}(nh) \lim_{x\to\infty} \left[\frac{\bar{G}(tx/((n+1)h)) - \bar{G}(tx/nh)}{\bar{G}(x)}\right] \le M_2. \sum_{n=1}^{n_0} \bar{F}(nh) < \infty.$$

Where the second inequality follows by

$$M_2 = \lim_{x\to\infty} \left[\frac{\bar{G}(tx/((n+1)h)) - \bar{G}(tx/nh)}{\bar{G}(x)}\right] < \infty. \qquad (by \ \ G \in D)$$

Furthermore, we have

$$\begin{aligned} \lim_{x\to\infty} K_4 &\le \lim_{x\to\infty} \sum_{n=n_0}^{N_0} \frac{\bar{F}(tx)}{\bar{G}(x)} \left[\bar{G}(tx/((n+1)h)) - \bar{G}(tx/nh)\right] \\ &\le M_2. \lim_{x\to\infty} \frac{\bar{G}(tx)}{\bar{G}(x)} [\bar{G}(tx/((N_0+1)h)) - \bar{G}(tx/n_0h)] < \infty. \end{aligned} \tag{2.8}$$

Where, last inequality follows from $G \in D$ and $M_2 = \lim_{x\to\infty} \bar{F}(tx)/\bar{G}(tx)$. Now by substituting (2.7), (2.8) and (2.8) in (2.6), proof completes. □

Corollary 2.3. *Let $X_1, ..., X_n$ be WND random variables with common distribution function $F \in D$. If $E(X) < \infty$ then $P_n = \prod_{i=1}^{n} X_i \in D$.*

Theorem 2.4. *Let X and Y be two WND random variables with distribution functions F and G, respectively. If $F, G \in D \cap L$ and $\bar{G}^*(t) < 1$ $(\bar{F}^*(t) < 1)$, then $H \in D \cap L$.*

Proof. By the same approach as used in the proof of Theorem 2.1, we have:

i. If $X > 1$ *a.s* and $Y > 1$ *a.s.*, then for any $u > 0$ and for all $x > 1$, applying (2.1) and (2.2), we have $1 \leq \bar{H}(x-u)/\bar{H}(x) \leq \max\{J_1, J_2\}$, where by using Remark 1.2,

$$J_1 = \frac{P(X > x-u) - P(X > x-u; Y > x-u)}{P(X > x) - P(X > x; Y > x)} \leq \frac{P(X > x-u)}{P(X > x) - C.P(X > x)P(Y > x)}.$$

Since $F \in L$, then $\lim_{x\to\infty} J_1 \leq 1$. For J_2 we have

$$J_2 \leq \frac{1}{\bar{G}(x)}[\bar{G}(x-u) + P(XY > x-u; X < x-u; Y < x-u)] = J_3 + J_4,$$

where $J_3 = \bar{G}(x-u)/\bar{G}(x)$ and

$$J_4 \leq C. \int_0^{x-u} \frac{\bar{G}((x-u)/t) - \bar{G}(x-u)}{\bar{G}(x)} dF(u).$$

Now, for each $t > 0$, $G \in L$ and $\bar{G}^*(t) < 1$ we have $\lim_{x\to\infty} J_3 = 1$. Also, by Remark 1.2,

$$\limsup_{x\to\infty} J_4 \leq C. \int_0^{\infty} \limsup_{x\to\infty} \frac{I_{(0,x-u)}(t)[\bar{G}((x-u)/t) - \bar{G}(x-u)]}{\bar{G}(x)} dF(t) \leq 0.$$

So, for each $u > 0$,

$$\lim_{x\to\infty} \frac{\bar{H}(x-u)}{\bar{H}(x)} = 1. \tag{2.9}$$

ii. If $X > 1$ *a.s.* and $0 < Y < 1$ *a.s.*, then

$$1 \leq \frac{P(XY > x-u)}{P(XY > x)} = 1 + \frac{P(x-u < XY < x)}{P(XY > x)} = 1 + J_5.$$

By Remark 1.2 and $F \in D \cap L$ we have,

$$\lim_{x\to\infty} J_5 \leq \lim_{x\to\infty} \frac{P(x-u < XY < x)}{P(X > x)} \leq \lim_{x\to\infty} C. \int_0^1 \frac{\bar{F}((x-u)/t) - \bar{F}(x/t)}{\bar{F}(x)} dG(t) = 0.$$

Hence,

$$\lim_{x\to\infty} \frac{\bar{H}(x-u)}{\bar{H}(x)} = 1. \tag{2.10}$$

iii. If $Y > 1$ *a.s.* and $0 < X < 1$ *a.s.*, then, similar to (*ii*) we can obtain (2.10).
Combining (2.2), (2.9) and (2.10), we derive

$$\begin{aligned} 1 \quad \leq \quad & \lim_{x\to\infty} \frac{\bar{H}(x-u)}{\bar{H}(x)} = \lim_{x\to\infty} \max\{\frac{P(XY > x-u; X > 1; Y > 1)}{P(XY > x; X > 1; Y > 1)} \\ & , \frac{P(XY > x-u; X < 1; Y > 1)}{P(XY > x; X < 1; Y > 1)}, \frac{P(XY > x-u; X > 1; Y < 1)}{P(XY > x; X > 1; Y < 1)}\} \leq 1 \end{aligned}$$

This completes the proof. □

Theorem 2.5. *Let Y_1 and Y_2 be two WND random variables with common distribution function $G \in L$ and $E(Y) < \infty$. Suppose that X_1 and X_2 are two independent random variables which are independent of Y_1 and Y_2, with distribution functions F_1 and F_2, respectively, then*

$$P(X_1Y_1 + X_2Y_2 > x) \sim P(X_1Y_1 > x) + P(X_2Y_2 > x) \quad as \quad x \to \infty.$$

Proof. For every $x > 0$, we have

$$\begin{aligned} P(X_1Y_1 + X_2Y_2 > x) &= \int_0^\infty \int_0^\infty P(x_1Y_1 + x_2Y_2 > x | X_1 = x_1, X_2 = x_2) dF_1(x_1) dF_2(x_2) \\ &\sim \int_0^\infty \int_0^\infty [P(x_1Y_1 > x) + P(x_2Y_2 > x)] dF_1(x_1) dF_2(x_2) \quad (as \quad x \to \infty) \\ &= P(X_1Y_1 > x) + P(X_2Y_2 > x). \end{aligned}$$

The asymptotic relation follows by Theorem 2 of Ranjbar, et al.[8], and this completes the proof. □

Conclusions: All Theorems and Lemmas are valid for $C = 1$, as a matter of fact, the independence structure is special case of our work.

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