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Approximate isometries in Hilbert C^*-modules *

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Abstract. We use the fixed point alternative theorem to prove that, under suitable conditions, every A-valued approximate isometry on a Hilbert C^*-module over the C^*-algebra A can be approximated by a unique A-valued isometry.

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1. Introduction

Let X and Y be Banach spaces and $\varepsilon > 0$. A mapping $f : X \to Y$ is called an approximate isometry, or an ε-isometry if $|\ ||f(x) - f(y)|| - ||x - y||\ | < \varepsilon$ for all $x, y \in X$. The question is whether $\varepsilon > 0$ being small implies the existence of an isometry I which is close to f. If the answer is affirmative, then we say that isometries are stable in the sense of Ulam. The first positive answer toward the solution of the problem was given by Hyers and Ulam [12] who proved that each ε-isometry from a Hilbert space onto itself can be uniformly approximated by an isometry within a distance of $k\varepsilon$, for some $k > 0$. In [13], they also solved the problem for mappings on the space of continuous functions over a compact metric space. In 1983, Gevirtz [10] showed that every surjective ε-isometry $f : X \to Y$ with $f(0) = 0$, where X and Y are real Banach spaces, can be uniformly approximated by a linear isometry U such that $||f(x) - U(x)|| \leq 5\varepsilon$ for each $x \in X$. Omladič and Šemrl [18] showed that 2ε is a sharp constant in the latter inequality.

The problem of isometries and its variants have been considered by many authors from different points of view (see e. g. [1, 6, 7], [9]-[14],[19, 22] and the references therein).

Hilbert C^*-modules are objects like Hilbert spaces except that the inner product takes its values in a C^*-algebra. Let E and F be Hilbert C^*-modules over a C^*-algebra A. A mapping $U : E \to F$ is called an isometry (or an A-valued isometry) if

$$|U(x) - U(y)| = |x - y|$$

holds, where $|\cdot|$ denotes the A-valued norm.

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In this paper we consider the Hyers-Ulam stability for the above functional equation. In fact, we use the fixed point alternative theorem to show that if E and F are Hilbert C^*-modules over a C^*-algebra A and $f : E \to F$ satisfies the inequality

$$|| \, |f(x) - f(y)| \; - \; |x - y| \, || \leq \varphi(x, y) \quad (x, y \in E) \tag{1}$$

then under suitable conditions on $\varphi : E \times E \to [0, \infty)$, there is a unique isometric mapping $U : E \to F$, which suitably approximates f.

2. Preliminaries

In 1953, Kaplansky [15] initiated the notion of a Hilbert C^*-module as a generalization of a Hilbert space in which the inner product takes its values in a C^*-algebra. It turned out that the notion of Hilbert C^*-module has many applications [16].

Definition 1. *Let A be a C^*-algebra. An inner product A-module is a linear space E which is a right A-module with compatible scalar multiplication:*

$$\lambda(xa) = (\lambda x)a = x(\lambda a) \quad (\lambda \in \mathbb{C}, x \in E, a \in A),$$

together with a map

$$(x, y) \mapsto \langle x, y\rangle : E \times E \to A$$

such that for each $x, y, z \in E$, $\alpha, \beta \in \mathbb{C}$ and $a \in A$,

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$,

(ii) $\langle x, ya\rangle \; = \; \langle x, y\rangle \; a$,

(iii) $\langle x, y\rangle^* = \langle y, x\rangle$,

(iv) $\langle x, x\rangle \geq 0$ *and the equality holds if and only if* $x = 0$.

It is easy to observe that $||x|| = ||\langle x, x\rangle||^{\frac{1}{2}}$ defines a norm on E. An inner product A-module which is complete with respect to the norm $||x||$ is called a *Hilbert A-module* or a *Hilbert C^*-module over the C^*-algebra A*. The notion of a left Hilbert A-module can be defined similarly. By (iv), $\langle x, x\rangle$ is a positive element of the C^*-algebra A, hence the *A-valued norm* $|x| = \langle x, x\rangle^{\frac{1}{2}}$ exists. The A-valued norm $| \, . \, |$ is a useful device but it needs to be handled with care. For example, it need not be the case that $|x + y| \leq |x| + |y|$ for each $x, y \in E$.

Example 1. *Every C^*-algebra A is a Hilbert A-module under the A-valued inner product*

$$\langle a, b\rangle = a^*b \qquad (a, b \in A).$$

Example 2. *Every complex Hilbert space is a left Hilbert $\mathbb{C}$-module.*

We refer the reader to [16] for basic properties of Hilbert C^*-modules.

Definition 2. *The pair (X, d) is called a generalized complete metric space if X is a nonempty set and $d : X^2 \to [0, \infty]$ satisfies the following conditions:*

(a) $d(x, y) \geq 0$ *and the equality holds if and only if* $x = y$*,*

(b) $d(x, y) = d(y, x)$*,*

(c) $d(x, z) \leq d(x, y) + d(y, z)$*,*

(d) every d*-Cauchy sequence in* X *is* d*-convergent.*

Note that the distance between two points in a generalized metric space is permitted to be infinity.

Definition 3. *Let* (X, d) *be a generalized complete metric space. A mapping* $\Lambda : X \to X$ *satisfies a Lipschitz condition with Lipschitz constant* $L \geq 0$ *if*

$$d(\Lambda(x), \Lambda(y)) \leq Ld(x, y) \quad (x, y \in X).$$

If $L < 1$*, then* Λ *is called a strictly contractive operator.*

In 2003, Radu [20] employed the following result, due to Diaz and Margolis [8], to prove the stability of a Cauchy functional equation. Using such an elegant idea, several authors applied the method to investigate the stability of some functional equations, see [2]-[5], [14, 17, 20, 21].

Proposition 1 (The fixed point alternative principle)**.** *Suppose that a complete generalized metric space* $(\mathcal{E}, d)$ *and a strictly contractive mapping* $J : \mathcal{E} \to \mathcal{E}$ *with the Lipschitz constant* $0 < L < 1$ *are given. Then, for a given element* $x \in \mathcal{E}$*, exactly one of the following assertions is true:*
either
(a) $d(J^n x, J^{n+1} x) = \infty$ *for all* $n \geq 0$ *or*
(b) there exists some integer k *such that* $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \geq k$*.*
Actually, if (b) holds, then the sequence $\{J^n x\}$ *is convergent to a fixed point* x^* *of* J *and*
(b1) x^* *is the unique fixed point of* J *in* $\mathcal{F} := \{y \in \mathcal{E}, d(J^k x, y) < \infty\}$*;*
(b2) $d(y, x^*) \leq \frac{d(y, Jy)}{1-L}$ *for all* $y \in \mathcal{F}$*.*

Remark 1. *The fixed point* x^**, if it exists, is not necessarily unique in the whole space* $\mathcal{E}$*; it may depend on* x*. Actually, if (b) holds, then* $(\mathcal{F}, d)$ *is a complete metric space and* $J(\mathcal{F}) \subset \mathcal{F}$*. Therefore properties (b1) and (b2) follow from "The Banach Fixed Point Theorem".*

3. Main results

Throughout the rest of the paper, we assume that E and F are Hilbert C^*-modules over a C^*-algebra A. A function $f : E \to F$ is said to be φ-approximately isometric if (1) holds.

Lemma 1. *Let* $\phi : E \times E \to [0, \infty)$ *and* $f : E \to F$ *be such that*

$$\| \, |f(x) - f(y)|^2 \; - \; |x - y|^2 \, \| \leq \phi(x, y) \quad (x, y \in E). \tag{2}$$

Suppose that

$$\lim_{n\to\infty} 2^{2n}\phi(2^{-n}x, 2^{-n}y) = 0 \quad (x, y \in E),$$

and for each $x \in E$,

$$2\psi(2^{-1}x) \le L\psi(x) \tag{3}$$

where $0 < L < 1$ and

$$\psi(x) = \Big(\phi(2x, 0) + 2\ \phi(x, 2x) + 2\ \phi(x, 0)\Big)^{\frac{1}{2}} \quad (x \in E). \tag{4}$$

Then there exists a unique isometry $U : E \to E$ such that

$$||f(x) - f(0) - U(x)|| \le \frac{L\psi(x)}{2(1-L)} \quad (x \in E) \tag{5}$$

and

$$U(2^{-n}x) = 2^{-n}U(x) \quad (x \in E, n \in \mathbb{N}). \tag{6}$$

Proof. By replacing f by $f_1 = f - f(0)$, we can assume that $f(0) = 0$. By substituting y by 0 and $2x$ in (2), we see that

$$||\ |f(x)|^2 - |x|^2|| \le \phi(x, 0) \tag{7}$$

and

$$||\ \ |f(2x) - f(x)|^2 - |x|^2\ || \le \phi(x, 2x) \tag{8}$$

for each $x \in E$. By replacing x by $2x$ and y by 0 in (2), we see that

$$||\ |f(2x)|^2 - 4|x|^2\ || \le \phi(2x, 0) \quad (x \in E). \tag{9}$$

Since

$$\begin{aligned} |f(x) - f(2x)|^2 &= \langle f(x) - f(2x), f(x) - f(2x)\rangle \\ &= |f(2x)|^2 + |f(x)|^2 - \big(\langle f(x), f(2x)\rangle + \langle f(2x), f(x)\rangle\big) \quad (x \in E), \end{aligned}$$

we have

$$\begin{aligned} |2f(x) - f(2x)|^2 &= \langle 2f(x) - f(2x), 2f(x) - f(2x)\rangle \\ &= 4|f(x)|^2 + |f(2x)|^2 - 2\big(\langle f(x), f(2x)\rangle + \langle f(2x), f(x)\rangle\big) \\ &= 2|f(2x) - f(x)|^2 + 2|f(x)|^2 - |f(2x)|^2 \\ &= 2(|f(2x) - f(x)|^2 - |x|^2) \\ &\quad +2(|f(x)|^2 - |x|^2) - (|f(2x)|^2 - 4|x|^2) \quad (x \in E). \end{aligned}$$

Therefore by (7), (8) and (9), we have

$$\begin{aligned} ||2f(x) - f(2x)||^2 &= ||\ |2f(x) - f(2x)|^2\ || \\ &\le 2||\ |f(2x) - f(x)|^2 - |x|^2\ || + 2||\ |f(x)|^2 - |x|^2\ || \\ &\quad +||\ |f(2x)|^2 - 4|x|^2\ || \\ &\le 2\ \phi(x, 2x) + 2\ \phi(x, 0) + \phi(2x, 0) \quad (x \in E). \end{aligned}$$

Hence

$$||2f(x) - f(2x)|| \leq \psi(x) \quad (x \in E). \tag{10}$$

Define

$$\mathcal{E} = \{g : E \to F : g(0) = 0\}$$

and

$$d(g,h) = \inf\{\alpha \geq 0 : ||g(x) - h(x)|| \leq \alpha\psi(x), \forall x \in E\} \quad (g,h \in \mathcal{E}).$$

Then $(\mathcal{E}, d)$ is a complete generalized metric space (see [20] for details). Let $J : \mathcal{E} \to \mathcal{E}$ be defined by $J(g)(x) = 2g(2^{-1}x)$ for each $x \in E$. We have

$$||g(x) - h(x)|| \leq d(g,h)\psi(x) \quad (x \in E).$$

Therefore for each $x \in E$,

$$||J(g)(x) - J(h)(x)|| = ||2g(2^{-1}x) - 2h(2^{-1}x)|| \leq 2d(g,h)\psi(2^{-1}x) \leq Ld(g,h)\psi(x).$$

Hence

$$d\big(J(g), J(h)\big) \leq Ld(g,h) \quad (g,h \in \mathcal{E}).$$

Therefore $J : \mathcal{E} \to \mathcal{E}$ is a strictly contractive mapping with the Lipschitz constant $0 < L < 1$. By (3) and (10), $||J(f)(x) - f(x)|| \leq \frac{L}{2}\psi(x)$ for each $x \in E$. This means that $d(J(f), f) \leq \frac{L}{2}$. Therefore, by Proposition 1, J has a unique fixed point $U : E \to F$ in the set $\mathfrak{F} = \{g \in \mathcal{E} : d(J(f), g) < \infty\}$ which is defined by $U(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$ for each $x \in E$ and

$$d(U,f) \leq \frac{d(f, J(f))}{1-L} \leq \frac{L}{2(1-L)}.$$

Hence, by the definition of d, (5) holds. We have

$$\begin{aligned} U(2^{-n}x) = \lim_{m\to\infty} 2^m f(2^{-(n+m)}x) &= \lim_{k\to\infty} 2^{-n}2^k f(2^{-k}x) \\ &= 2^{-n}U(x) \quad (x \in E, n \in \mathbb{N}). \end{aligned}$$

This proves (6). Since

$$\begin{aligned} ||\ |U(x) - U(y)|^2 - |x-y|^2\ || &= \lim_{n\to\infty} ||\ |2^n f(2^{-n}x) - 2^n f(2^{-n}y)|^2 - |x-y|^2\ || \\ &\leq \lim_{n\to\infty} 2^{2n}\phi(2^{-n}x, 2^{-n}y) = 0 \quad (x,y \in E), \end{aligned}$$

we have

$$|U(x) - U(y)| = |x - y| \quad (x,y \in E).$$

Let $U' : E \to F$ be an isometry such that

$$||f(x) - f(0) - U'(x)|| \leq \frac{L\psi(x)}{2(1-L)} \quad (x \in E),$$

and

$$U'(2^{-n}x) = 2^{-n}U'(x) \quad (x \in E, n \in \mathbb{N}).$$

Then by the triangle inequality

$$||U(x) - U'(x)|| \leq \frac{L\psi(x)}{1-L} \quad (x \in E).$$

Therefore for each $x \in E$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} ||U(x) - U'(x)|| &= 2^n||U(2^{-n}x) - U'(2^{-n}x)|| \\ &\leq \frac{L2^n\psi(2^{-n}x)}{1-L} \\ &\leq \frac{L^{n+1}\psi(x)}{1-L} \quad (n \in \mathbb{N}). \end{aligned}$$

Since $L < 1$, the last term of the above inequality tends to zero as n tends to infinity. This proves the uniqueness in the assertion of Lemma 1. □

Theorem 1. *Let $f : E \to F$ be a φ-approximately isometric function for some function $\varphi : E \times E \to [0, \infty)$ such that*

$$\lim_{n\to\infty} 2^n\varphi(2^{-n}x, 2^{-n}y) = 0 \quad (x, y \in E). \tag{11}$$

Define

$$\phi(x, y) = \varphi(x, y)\Big(\varphi(x, y) + 2||x - y||\Big) \quad (x, y \in E). \tag{12}$$

Let ψ be defined by (4) *and assume that* (3) *holds for some $0 < L < 1$. Then there exists a unique isometry $U : E \to F$ such that* (5) *and* (6) *hold.*

Proof. For every $x, y \in E$, we have

$$\begin{aligned} |f(x) - f(y)|^2 \ - \ |x - y|^2 &= |f(x) - f(y)|\Big(|f(x) - f(y)| - |x - y|\Big) \\ &\quad + \Big(|f(x) - f(y)| - |x - y|\Big)|x - y|. \end{aligned}$$

Since by (1)

$$||f(x) - f(y)|| \leq \varphi(x, y) + ||x - y|| \quad (x, y \in E),$$

we have

$$||\ |f(x) - f(y)|^2 \ - \ |x - y|^2\ || \leq \varphi(x, y)\Big(\varphi(x, y) + 2||x - y||\Big) \quad (x, y \in E).$$

Since

$$2^{2n}\phi(2^{-n}x, 2^{-n}y) = 2^n\varphi(2^{-n}x, 2^{-n}y)\Big(2^n\varphi(2^{-n}x, 2^{-n}y) + 2||x - y||\Big)$$
$$(x, y \in E; n \in \mathbb{N})$$

by (11), $\lim_{n\to\infty} 2^{2n}\phi(2^{-n}x, 2^{-n}y) = 0$ for each $x, y \in E$. Hence, by Lemma 1, there exists a unique isometry $U : E \to F$ such that (5) and (6) hold. □

Corollary 1. *Let $f : E \to F$ be a function such that for some $p > 1$,*

$$|| \, |f(x) - f(y)| - |x - y| \, || \le ||x - y||^p \quad (x, y \in E).$$

Then there exists a unique isometry $U : E \to F$ such that (6) and the approximation

$$||f(x) - f(0) - U(x)|| \le \frac{\Big((2^{2p} + 4)||x||^{2p} + (2^{p+2} + 8)||x||^{p+1}\Big)^{\frac{1}{2}}}{2^{\frac{p+1}{2}} - 2} \quad (x \in E) \qquad (13)$$

hold.

Proof. Let

$$\varphi(x, y) = ||x - y||^p \quad (x, y \in E).$$

Then

$$\lim_{n\to\infty} 2^n \varphi(2^{-n}x, 2^{-n}y) = \lim_{n\to\infty} 2^{n(1-p)}||x - y||^p = 0 \quad (x, y \in E)$$

and the functions ϕ and ψ which are defined by (12) and (4) are

$$\phi(x, y) = ||x - y||^{2p} + 2||x - y||^{p+1} \quad (x, y \in E)$$

and

$$\psi(x) = \Big((2^{2p} + 4)||x||^{2p} + (2^{p+2} + 8)||x||^{p+1}\Big)^{\frac{1}{2}} \quad (x \in E),$$

respectively. We have

$$\begin{aligned} 2\psi(2^{-1}x) &= 2\Big(2^{-2p}(2^{2p} + 4)||x||^{2p} + 2^{-(p+1)}(2^{p+2} + 8)||x||^{p+1}\Big)^{\frac{1}{2}} \\ &\le 2 \max\{2^{-p}, 2^{\frac{-(p+1)}{2}}\}\psi(x) \\ &= 2^{\frac{-(p+1)}{2}}\psi(x) \quad (x \in E). \end{aligned}$$

Hence for $L = 2^{\frac{-(p+1)}{2}} < 1$, (3) holds. By Theorem 1, there is a unique isometry $U : E \to F$ such that (6) and (13) hold. □

The proof of the following result is similar to the proof of Lemma 1, hence it is omitted.

Lemma 2. *Let $\phi : E \times E \to [0, \infty)$ be a mapping and $f : E \to F$ satisfy (2). If*

$$\lim_{n\to\infty} 2^{-2n}\phi(2^n x, 2^n y) = 0 \quad (x, y \in E),$$

and for each $x \in E$,

$$\frac{1}{2}\psi(2x) \le L\psi(x), \qquad (14)$$

where $0 < L < 1$ and ψ is defined by (4). Then there exists a unique isometry $U : E \to F$ such that

$$||f(x) - f(0) - U(x)|| \le \frac{\psi(x)}{2(1 - L)} \quad (x \in E) \qquad (15)$$

and

$$U(2^n x) = 2^n U(x) \quad (x \in E). \qquad (16)$$

By applying Lemma 2 and imitating the proof of Theorem 1, one can easily prove the following result.

Theorem 2. *Let $\varphi : E \times E \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} 2^{-n}\varphi(2^n x, 2^n y) = 0 \quad (x, y \in E).$$

Let $f : E \to F$ be a φ-approximately isometric mapping, let ϕ and ψ be defined by (12) and (4), respectively. Suppose that for each $x \in E$, inequality (14) holds. Then there exists a unique isometry $U : E \to F$ such that (15) and (16) hold.

In particular, we have:

Corollary 2. *Let $\varepsilon > 0$ and $f : E \to F$ satisfy the inequality*

$$||\ |f(x) - f(y)| - |x - y|\ || \le \varepsilon \quad (x, y \in E).$$

Then there exists a unique isometry $U : E \to F$ such that (16) and

$$||f(x) - f(0) - U(x)|| \le \frac{\Big(\varepsilon(5\varepsilon + 12||x||)\Big)^{\frac{1}{2}}}{2-\sqrt{2}} \quad (x \in E) \tag{17}$$

hold.

Proof. Let

$$\varphi(x, y) = \varepsilon \quad (x, y \in E).$$

Then

$$\lim_{n\to\infty} 2^{-n}\varphi(2^n x, 2^n y) = \lim_{n\to\infty} 2^{-n}\varepsilon = 0 \quad (x, y \in E).$$

Let ϕ be defined by (12), then

$$\phi(x, y) = \varepsilon\Big(\varepsilon + 2||x - y||\Big) \quad (x, y \in E).$$

It is not hard to see that the function ψ defined by (4), is

$$\psi(x) = \Big(\varepsilon(5\varepsilon + 12||x||)\Big)^{\frac{1}{2}} \quad (x \in E).$$

Since

$$2^{-1}\psi(2x) = 2^{-1}\Big(\varepsilon(5\varepsilon + 24||x|)\Big)^{\frac{1}{2}} \le 2^{-1}\sqrt{2}\psi(x) \quad (x \in E),$$

for $L = \frac{1}{\sqrt{2}} < 1$, all conditions of Theorem 2 are fulfilled. Therefore there exists a unique isometry $U : E \to F$ such that (16) and (17) hold. □

Hyers and Ulam [12] gave an example of an ε-isometry $f : \mathbb{R} \to \mathbb{R}^2$ such that $\{||f(x) - U(x)|| : x \in \mathbb{R}\}$ is an unbounded set for every isometry $U : \mathbb{R} \to \mathbb{R}^2$. Here we give such an example in the setting of complex Hilbert spaces.

Example 3. *Let $E = \mathbb{C}$, $F = \mathbb{C}^2$ and $\varepsilon > 0$. Then E and F are $\mathbb{C}$-modules. Define $f : E \to F$ by $f(z) = (z, \varepsilon + \sqrt{2\ \varepsilon|z|})$. Then for each $z, w \in E$, we have*

$$|z-w| \le \left\{|z-w|^2 + (\sqrt{2\ \varepsilon|z|} - \sqrt{2\ \varepsilon|w|})^2\right\}^{\frac{1}{2}} = |f(z) - f(w)|. \tag{18}$$

Without loss of generality, we may assume that $|z| \ge |w|$. Then $\sqrt{|z||w|} \ge |w|$, so that

$$|\sqrt{|z|} - \sqrt{|w|}|^2 = |z| + |w| - 2\sqrt{|z||w|} \le |z| + |w| - 2|w| = \ |z| - |w|\ \le |z-w|.$$

Hence by (18)*, we have*

$$\begin{aligned} |z-w| \le |f(z) - f(w)| &\le \left\{|z-w|^2 + 2\ \varepsilon|z-w|\right\}^{\frac{1}{2}} \\ &< \left\{|z-w|^2 + 2\ \varepsilon|z-w| + \varepsilon^2\right\}^{\frac{1}{2}} \\ &= |z-w| + \varepsilon. \end{aligned}$$

This shows that f is an ε-isometry. By Corollary 2, $U(z) = \lim_{n\to\infty} 2^{-n} f(2^n z) = (z, 0)$ is the unique suitable isometry corresponding to f. However,

$$d(f, U) = \sup\{|f(z) - U(z)| : z \in E\} = \sup\{\varepsilon + \sqrt{2\varepsilon|z|} : z \in E\} = \infty.$$

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