ENO Scheme for Steady and Transient Compressible Flow in Pressure-Based Method

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ABSTRACT

In this paper, a scheme based on Essentially Nonoscillating (ENO) has been developed into an implicit finite volume procedure, which uses pressure as a working variable. The boundedness criteria determined from ENO schemes. This numerical process is used for solution of Euler equations on a nonorthogonal mesh with collocated finite volume formulation. The developed scheme is applied to the computation of steady subsonic and transonic flows over a bump-in-channel geometry as well as to the transient shock-tube problem. The results of the ENO scheme are compared with analytical and other computations published in literature.

Keywords**:** Finite Volume, ENO, Transient

INTRODUCTION

The numerical solution of the transonic and supersonic flows is usually carried out by time-marching schemes that solve the set of the coupled system of equations governing the flux of mass, momentum, and energy, using accurate high-resolution total variation diminishing (TVD) [1] scheme employing Roe's Riemann solver[2]. On the other hand, simulation of incompressible fluid flows with engineering interest are usually pursued with finite volume formulation, using primitive variables in conjunction with some variant of the semi-implicit pressure-correction method [3]. In this method the momentum equations are solved in a segregated fashion while an equation for the pressure field or pressure-correction field is derived combining the discrete momentum and continuity equations so that the pressure field is driven towards a level where the continuity equation over each control volume is satisfied to any prescribed level [3-5]. Because this procedure in its standard form results in an elliptic equation for pressure (or pressure correction) it cannot cope with the hyperbolic nature of the signal propagation in compressible transonic or supersonic flows. By contrast those methods that are very efficient for computation of compressible hyperbolic flows become increasingly ill conditioned as Mach number decreases. Although some remedy for convergence stagnation exists such as artificial compressibility or preconditioning, in practice, these techniques are not well suited for computations of flows with extensive regains of low Mach numbers.

Several attempts have been made by incompressible fluid flow numerical researchers, towards the unification of numerical methods developed for incompressible and compressible flows. The main goal consists in the development of methods for computation of flows at all Mach numbers by extending the pressure-correction formulation to ensure shock-capturing properties. Leonard [6] has generalized the formulation of the high-resolution flux limiter schemes using what is called the normalized variable formulation (NVF). Many schemes based on the NVF has been developed in pressure-based method, for example SMART scheme[7], SFCD scheme[8], SOUCUP scheme[9], STOIC scheme[10], SBIC scheme base on variable and flux limiter[11,12]. Issa and Javareshkian [13] implemented a high resolution TVD scheme with characteristic-variables-based flux limiters into a pressure-based finite volume method. Batten et al.[14] utilized the TVD approach and adopted a time marching technique. The TVD and NVD schemes do not have oscillation at discontinuities because they are switched to first order scheme. The ENO scheme is presented for the first time by Harten *et. al* [15]. The ENO scheme do not have a Gibbs-like phenomenon O(1) at discontinuities, yet they may occasionally produce small spurious oscillations on the level O(h^r) of the truncation error. Kobayashi and Pereira [11] introduced an ENO scheme into pressure-correction solution procedures for the flux calculation, which they incorporated into a steady-state solution method.

The objection of this paper is to extend an Essentially Non-oscillating (ENO) scheme to the computation of steady-state and transient flows in pressure-based method. The developed scheme is applied to the computation of steady subsonic and transonic flows over a bump-in-channel geometry as well as to the transient shock-tube problem. The results of the ENO scheme are compared with other computations published in literatures.

GOVERNING EQUATION

The basic equations, which describe conservation of mass, momentum, and scalar quantities, can be expressed in Cartesian tensor form as:

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u_j)}{\partial x_j}=0 \tag{1}$$

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial(\rho u_i u_j - T_{ij})}{\partial x_j}=S_i^u \tag{2}$$

$$\frac{\partial(\rho\phi)}{\partial t}+\frac{\partial(\rho u_j \phi - q_j)}{\partial x_j}=S^\phi \tag{3}$$

The stress tensor and scalar flux vector are usually expressed in terms of basic dependent variables. The stress tensor for a Newtonian fluid is:

$$T_{ij} = -p\delta_{ij} - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{4}$$

The scalar flux vector is usually given by the Fourier-type law:

$$q_j = \Gamma_\phi\left(\frac{\partial\phi}{\partial x_j}\right) \tag{5}$$

DISCRETIZATION

The discretizations of the above differential equations are carried out using a finite-volume approach. First, the solution domain is divided into a finite number of discrete volumes or "cells ", where all variables are stored at their geometric centers (see e.g. Fig.1). The equations are then integrated over all the control volumes by using the Gaussian theorem. The development of the discrete expressions to be presented is affected with reference to only one face of the control volume, namely, e, for the sake of brevity. For any variable ϕ (which may now also stand for the velocity components), the result of the integration yields:

$$\frac{\delta\upsilon}{\delta t}[(\rho\phi)_p^{n+1}-(\rho\phi)_p^n]+I_e-I_w+I_n-I_s=S_\phi\,\delta\upsilon \tag{6}$$

Where I 's are the combined cell-face convection I^c and diffusion I^D fluxes. The diffusion flux is approximated by central differences and can be written for cell-face *e* of the control volume in Fig.(1) as:

$$\mathrm{I_e^D} = \mathrm{D_e}(\phi_\mathrm{p}-\phi_\mathrm{E})-\mathrm{S_e^\phi} \tag{7}$$

Where S_e^ϕ stands for cross derivative which arising from mesh non-orthogonality. The discretization of the convective flux, however, requires special attention and is the subject of the various schemes developed. A representation of the convective flux for cell-face 'e' is:

$$I_e^c = (\rho.V.\mathrm{A})_e\phi_e = F_e\phi_e \tag{8}$$

The expression for the I_e by the ENO scheme is dealt with later. The discretized equations resulting from each approximation take the form:

$$a_p.\phi_p = \sum_{m=E,W,N,S} a_m.\phi_m + S'_\phi \tag{9}$$

where a's are the convection-diffusion coefficients.

CONVECTIVE FLUXES

The expression for the mass, momentum, and energy fluxes in Eq.(8) are determined by an essentially non-oscillatory (ENO) scheme. To evaluate flux, the second-order accurate piecewise linear reconstruction of the Riemann variable is applied on each control volume followed by solution of the Riemann problem. A piecewise polynomial is supposed by the distribution of the Riemann variable inside each control volume, and the limiters are applied on the Riemann variable gradients. In order to calculate the convective fluxes on the left and right of cell-face, the conservative variables are calculated on cell-faces. Then the flux at the interface by solving the Riemann problem with right and left values is evaluated. The characteristic variables and Roe's approximate Riemann solver are used for computation of the inviscid flux. In general, the expression can be written as:

$$I^c(u(0)) = \frac{1}{2}[I^c(u_R) + I^c(u_L) - \sum_{i=1}^{4} r_i \left|\lambda_i\right| \Delta W_i] \tag{10}$$

Where I^c is a mass, momentum and energy fluxes. r_i and λ_i are right eigenvectors and eigenvalues of $\frac{\partial I(u)}{\partial u}$ respectively and $\Delta w_i = L_i(u_R - u_L)$. L_i are the left eigenvectors of $\frac{\partial I(u)}{\partial u}$ and u can be defined as:

$$u = (\rho, \rho u, \rho v, \rho e_T) \tag{11}$$

u_L and u_R can be calculated in a same approach. As an example, u_L for the crosses face *e* is calculated as follows:

$$u_L = u_p + \frac{1}{2}\sum_{i=1}^{4} r_{i,P}\Delta w_{i,P} \tag{12}$$

$$\Delta w_{i,P} = M(\alpha^-_{i,P}, \alpha^+_{i,P}) \tag{13}$$

$$\alpha^-_{i,P} = \Delta w_{i,w} + \frac{1}{2}\beta_{i,w} \tag{14}$$

$$\alpha^+_{i,P} = \Delta w_{i,e} - \frac{1}{2}\beta_{i,e} \tag{15}$$

$$\begin{aligned} \beta_{i,e} &= M(\beta_{i,P}, \beta_{i,E}) \\ \beta_{i,w} &= M(\beta_{i,W}, \beta_{i,P}) \end{aligned} \tag{16}$$

$$\begin{aligned} \beta_{i,W} &= \Delta w_{i,w} - \Delta w_{i,ww} \\ \beta_{i,P} &= \Delta w_{i,e} - \Delta w_{i,w} \\ \beta_{i,E} &= \Delta w_{i,ee} - \Delta w_{i,e} \end{aligned} \tag{17}$$

$$\begin{aligned} \Delta w_{i,ww} &= L_i(u_W - u_{WW}) \\ \Delta w_{i,w} &= L_i(u_P - u_W) \\ \Delta w_{i,e} &= L_i(u_E - u_P) \\ \Delta w_{i,ee} &= L_i(u_{EE} - u_E) \end{aligned} \tag{18}$$

By Roe average method, L_i are calculated in direction of normal vector of *e* surface for interested cells. M stands for the min mod function:

$$M(a,b) = \frac{(sign(a) + sign(b))}{2} \min(|a|,|b|) \qquad (19)$$

Where $sign(a)$ is using function. The same approach is used for other sides (*w*, *n*, s) of cells.

SOLUTION ALGORITHM

Most contemporary pressure-based methods employ a sequential iteration technique in which the different conservation equations are solved one after another. The common approach taken in enforcing continuity is by combining the equation for continuity with those of momentum to derive an equation for pressure or pressure-correction. The PISO algorithm is used in this work.

RESULTS

Both two-dimensional steady and one-dimensional transient flows are computed and the results are compared either with existing numerical solutions obtained by others or with the analytic solutions when they are available. The test cases chosen are the normal benchmarks to which methods such as the one presented here are applied. The first case is that of the classical shock tube problem and the second is the bump-in-channel case.

Fig. 2 shows the spatial distribution of velocity, density, Mach number and pressure ratio, along the shock tube at a given instant in time in a shock-tube for an initial pressure of 10. The results of computation on a mesh of 100 nodes are compared with the analytic solution. It can be seen that the shock is sharply captured, and the contact discontinuity is better resolved and oscillation is not relatively produced for the ENO scheme.

The second case is transonic flow over 10% thick bump on a channel wall. For this test, stagnation pressure P_o, stagnation temperature T_o and the inlet angle are specified. At the outlet, the static pressure is fixed for subsonic outlet flows. Slip boundary conditions are used on the upper and lower walls. A non-uniform grid of 98×26 in which the grid lines are closely packed in and near the bump region is shown in Fig.3.

The results of transonic flow with inlet Mach number equal to 1.4 over a 10% thick bump are shown in Fig. 4. The pressure ratio distribution on the upper and lower surfaces for present scheme are compared with the TVD [12] prediction. The agreement between the two solutions is remarkable, thus once again verifying the validity of the ENO scheme in pressure-based algorithm.

CONCLUSION

A pressure-based implicit procedure is described. It incorporates a ENO scheme. The developed scheme is applied to both transient and steady state flows and the results are in well agreements with other schemes.

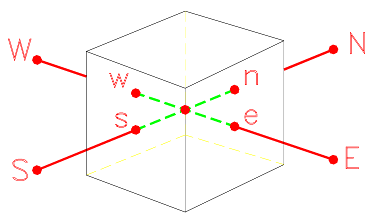
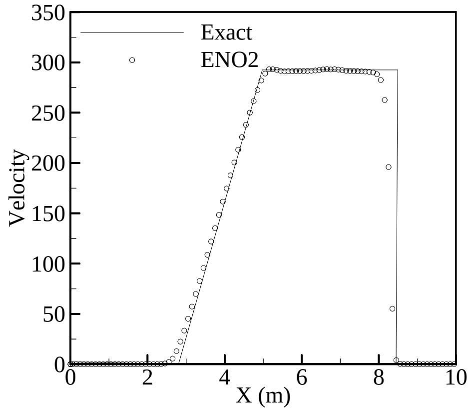
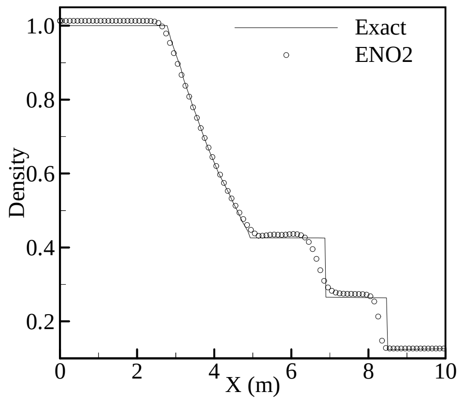


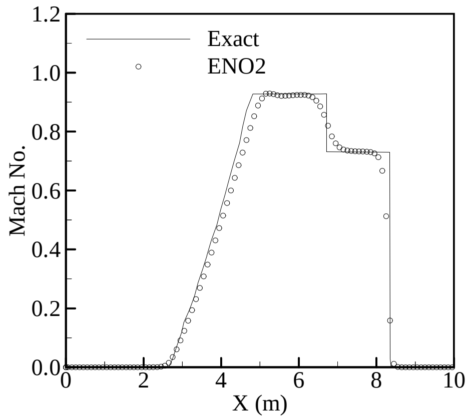
Fig.1, Finite volume and storage arrangement



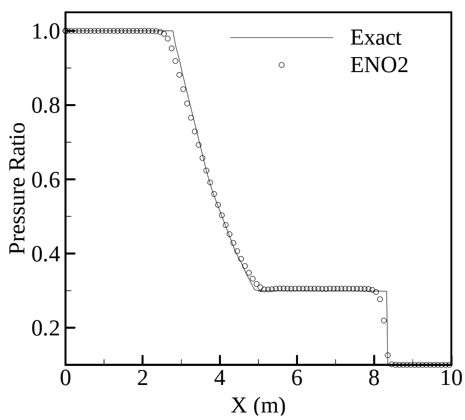
a) Velocity



b) Density



c) Mach Number



d) Pressure ratio distribution

Fig. 2, Shock-tube results for an initial pressure ratio $\frac{P_H}{P_L} = 10$ **at time** t_0 **=6.0**

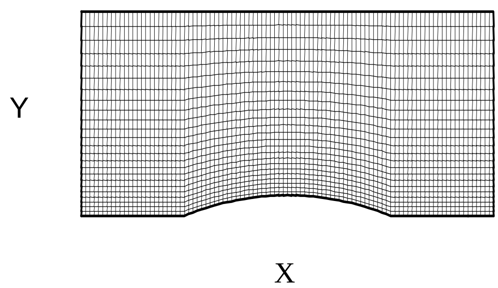


Fig. 3, Geometry

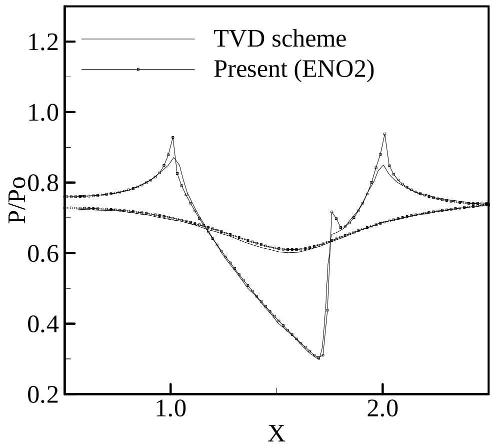


Fig. 4, Transonic flow over 10% thick bump, inlet M_{∞} =0.675

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