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δ-DOUBLE DERIVATIONS ON C^*-ALGEBRAS

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Abstract. Let $\mathcal{A}$ be an algebra and $\delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ be linear mappings. We say that a linear mapping $d : \mathcal{A} \to \mathcal{A}$ is a (δ, ε)-double derivation if $d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b)$ for all $a, b \in \mathcal{A}$. By a δ-double derivation we mean a (δ, δ)-double derivation. Giving some elementary facts concerning double derivations, we prove that if $\mathcal{A}$ is a C^*-algebra, $\delta : \mathcal{A} \to \mathcal{A}$ is a $*$-linear mapping and $d : \mathcal{A} \to \mathcal{A}$ is a continuous δ-double derivation then δ is continuous. We also show that if $\mathcal{A}$ is a C^*-algebra, $\delta : \mathcal{A} \to \mathcal{A}$ is a continuous linear mapping and $d : \mathcal{A} \to \mathcal{A}$ is a $*$-δ-double derivation then d is continuous. Similar facts concerning (δ, ε)-double derivations on C^*-algebras are also given.

1. **Introduction**

Let $\mathcal{A}$ be a subalgebra of an algebra $\mathcal{B}$, $\mathcal{X}$ be a $\mathcal{B}$-bimodule and $\sigma : \mathcal{A} \to \mathcal{B}$ be a linear mapping. A linear mapping $d : \mathcal{A} \to \mathcal{X}$ is called a σ-derivation (see [5] and [6]) if

$$d(ab) = d(a)\sigma(b) + \sigma(a)d(b) \quad a, b \in \mathcal{A}. \tag{1.1}$$

Clearly, if $\sigma = id$, the identity mapping on $\mathcal{A}$, then a σ-derivation is an ordinary derivation. On the other hand, each homomorphism d is a $\frac{d}{2}$-derivation. Thus, the theory of σ-derivations combines the theory of

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derivations and homomorphisms. If $\delta : \mathcal{A} \to \mathcal{A}$ is an ordinary derivation and $\sigma : \mathcal{A} \to \mathcal{A}$ is a homomorphism, then $d = \delta\sigma$ is a σ-derivation. Although, a σ-derivation is not necessarily of the form $\delta\sigma$, but it seems that the generalized Leibniz rule, $d(ab) = d(a)\sigma(b) + \sigma(a)d(b)$, comes from this observation. Taking ideas from this fact, we motivate to consider two derivations $\delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ to find a similar rule, for $d = \delta\varepsilon$. In this case, we see that d satisfies

$$d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b) \quad a, b \in \mathcal{A}. \tag{1.2}$$

Fortunately, this can be perceived as a generalization of the notion of a σ-derivation. We say that a linear mapping $d : \mathcal{A} \to \mathcal{A}$ is a (δ, ε)-double derivation if it satisfies (1.2).

The problem of automatic continuity of derivations is an important problem in the theory of derivations. In 1960, H. Sakai [11] proved that every derivation on a C^*-algebra is automatically continuous and later in 1972, J. R. Ringrose [10] showed that every derivation from a C^*-algebra into a Banach $\mathcal{A}$-bimodule is continuous. The problem of automatic continuity is also considered for σ-derivations. In 2006, M. Mirzavaziri and M. S. Moslehian [5] acquired some results about automatic continuity of σ-derivations. Suppose that $\mathcal{A}$ is a C^*-algebra acting on a Hilbert space $\mathcal{H}$. In [5], it is proved that if $\sigma : \mathcal{A} \to B(\mathcal{H})$ is a continuous $*$-linear mapping then every σ-derivation from $\mathcal{A}$ to $B(\mathcal{H})$ is automatically continuous. Moreover, the converse is established in [5] in the sense that if $d : \mathcal{A} \to B(\mathcal{H})$ is a continuous $*$-σ-derivation then there exists a continuous mapping $\Sigma : \mathcal{A} \to B(\mathcal{H})$ such that d is a $*$-Σ-derivation. Here, we consider the same problem for double derivations. Since the notion of a double derivation is a generalization of derivation, homomorphism and σ-derivation, our results extend the previous facts. Although our proof are similar to the previous arguments in some cases, but they are essentially new.

The reader is referred to [8] and [9] for the definitions and elementary properties of C^*-algebras, to [1],[2],[4],[5],[6] and [7] for various generalized notions of derivations and to [3],[10],[11] and [12] for more information on automatic continuity of derivations, inner derivations, point derivations and higher derivations.

2. **Preliminaries**

Definition 2.1. Let $\mathcal{A}$ be an algebra and $\delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ be linear mappings. A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is called a (δ, ε)-double derivation if

$$d(ab) = d(a)b + ad(b) + \delta(a)\varepsilon(b) + \varepsilon(a)\delta(b)$$

for all $a, b \in \mathcal{A}$. By a δ-double derivation we mean a (δ, δ)-double derivation.

It is clear that each σ-derivation $d : \mathcal{A} \to \mathcal{A}$ is a $(\sigma - id, d)$-double derivation. Moreover, every homomorphism $\varphi : \mathcal{A} \to \mathcal{A}$ is a $(\frac{\varphi}{2} - id, \varphi)$-double derivation.

Lemma 2.2. *If δ is a derivation on $\mathcal{A}$, then each δ-double derivation $d : \mathcal{A} \to \mathcal{A}$ is of the form $d = \delta^2 + \varepsilon$, where $\varepsilon : \mathcal{A} \to \mathcal{A}$ is a derivation.*

Proof. Let $\varepsilon = d - \delta^2$. Then,

$$\begin{aligned} \varepsilon(ab) &= (d - \delta^2)(ab) \\ &= d(a)b + ad(b) + 2\delta(a)\delta(b) - \delta(\delta(a)b + a\delta(b)) \\ &= d(a)b + ad(b) + 2\delta(a)\delta(b) - \delta^2(a)b - 2\delta(a)\delta(b) - a\delta^2(b) \\ &= (d - \delta^2)(a)b + a(d - \delta^2)(b) \\ &= \varepsilon(a)b + a\varepsilon(b). \end{aligned}$$

Hence ε is a derivation. □

Remark 2.3. Lemma 2.2 shows that if $\mathcal{A}$ is an algebra such that each derivation defined on $\mathcal{A}$ is automatically continuous and δ is a derivation then each δ-double derivation is also automatically continuous.

3. δ**-double derivations**

Recall that if $\mathcal{Y}$ and $\mathcal{Z}$ are normed spaces and $T : \mathcal{Y} \to \mathcal{Z}$ is a linear mapping, then the set of all z such that there is a sequence $\{y_n\}$ in $\mathcal{Y}$ with $y_n \to 0$ and $Ty_n \to z$ is called the separating space $\mathcal{S}(T)$ of T. Clearly, $\mathcal{S}(T) = \cap_{n=1}^{\infty} \overline{\{T(y) : \|y\| < 1/n\}}$ is a closed linear space. If $\mathcal{Y}$

and $\mathcal{Z}$ are Banach spaces, by the closed graph theorem, T is continuous if and only if $\mathcal{S}(T) = \{0\}$. Also, recall that if E is a subset of an algebra $\mathcal{B}$, the right annihilator $ran(E)$ of E (resp., the left annihilator $lan(E)$ of E) is defined to be $\{b \in \mathcal{B} : Eb = \{0\}\}$ (resp., $\{b \in \mathcal{B} : bE = \{0\}\}$). The set $ann(E) := ran(E) \cap lan(E)$ is called the annihilator of E.

Lemma 3.1. *Let $\mathcal{A}$ be a C^*-algebra acting on a Hilbert space $\mathcal{H}$. If $d : \mathcal{A} \to \mathcal{A}$ is a continuous δ-double derivation then $\mathcal{S}(\delta) \subseteq ann(\delta(\mathcal{A}))$.*

Proof. Let $A \in \mathcal{S}(\delta)$. Thus, there is a sequence $\{A_n\} \subseteq \mathcal{A}$ such that $A_n \to 0$ and $\delta(A_n) \to A$. For each $B \in \mathcal{A}$ we have,

$$\lim_{n\to\infty} d(A_nB) = \lim_{n\to\infty} A_n d(B) + \lim_{n\to\infty} d(A_n)B + \lim_{n\to\infty} 2\delta(A_n)\delta(B).$$

Since d is continuous, we obtain $0 = A\delta(B)$. Similarly, $0 = \delta(B)A$ and so $A \in ann(\delta(\mathcal{A}))$. □

Theorem 3.2. *Let $\mathcal{A}$ be a C^*-algebra acting on a Hilbert space $\mathcal{H}$, $\delta : \mathcal{A} \to \mathcal{A}$ be a $*$-linear mapping and $d : \mathcal{A} \to \mathcal{A}$ be a continuous δ-double derivation. Then, δ is continuous.*

Proof. Let d be continuous, $\mathcal{L}_0 = \bigcup_{A\in\mathcal{A}} \delta(A)(\mathcal{H})$ and $\mathcal{L}$ be the closed linear span of $\mathcal{L}_0$. Then, $\mathcal{H} = \mathcal{L} \oplus \mathcal{K}$, where $\mathcal{K} = \mathcal{L}^\perp$. By Lemma 3.2. in [4], we have,

$$\mathcal{K} = \bigcap_{A\in\mathcal{A}} \ker \delta(A).$$

Assume that $\{A_n\} \subseteq \mathcal{A}$, $A_n \to 0$ and $\delta(A_n) \to A$. Then, for each $\ell \in \mathcal{L}_0$ there is a $B \in \mathcal{A}$ and there is an $h \in \mathcal{H}$ such that $\ell = \delta(B)(h)$. Now, since $\mathcal{S}(\delta) \subseteq ann(\delta(\mathcal{A}))$, $A(\ell) = A(\delta(B)(h)) = (A\delta(B))(h) = 0$, then $A = 0$ on $\mathcal{L}_0$ and so $A = 0$ on $\mathcal{L}$. On the other hand,

$$A(k) = \lim_{n\to\infty} (\delta(A_n))(k) = 0.$$

Thus, $A = 0$ on $\mathcal{K}$ and so $A = 0$ on $\mathcal{H}$. Therefore, $\mathcal{S}(\delta) = \{0\}$ and δ is continuous. □

Theorem 3.3. *Let $\mathcal{A}$ be a C^*-algebra, $\delta : \mathcal{A} \to \mathcal{A}$ be a continuous linear mapping and $d : \mathcal{A} \to \mathcal{A}$ be a $*$-δ-double derivation. Then, d is continuous.*

Proof. We may assume that $\mathcal{A}$ is unital. In fact, if $\mathcal{A}$ has no identity, we shall consider the unitization $\mathcal{A}_1$ of $\mathcal{A}$ with unit 1 and define $d(1) = \delta(1) = 0$. Then, d and δ can be uniquely extended to linear mappings d_1 and δ_1 on $\mathcal{A}_1$. Moreover, d_1 is a δ_1-double derivation since

$$\begin{aligned} d_1[(a+\alpha)(b+\beta)] &= d_1[ab+\alpha b+a\beta+\alpha\beta] \\ &= d_1(ab)+\alpha d_1(b)+d_1(a)\beta+0 \\ &= ad_1(b)+d_1(a)b+2\delta_1(a)\delta_1(b)+\alpha d_1(b)+d_1(a)\beta \\ &= (a+\alpha)d_1(b)+d_1(a)(b+\beta)+2\delta_1(a)\delta_1(b) \\ &= (a+\alpha)d_1(b+\beta)+d_1(a+\alpha)(b+\beta)+2\delta_1(a+\alpha)\delta_1(b+\beta). \end{aligned}$$

Thus, it is sufficient to prove that a $*$-δ-double derivation d on a unital C^*-algebra $\mathcal{A}$ is continuous if so is δ. For this, we show that the $*$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ defined by $D(a) = d(a) - ad(1)$ for each $a \in \mathcal{A}$ is continuous. Suppose that δ is continuous and a is a self-adjoint element in $\mathcal{A}$. Also, let φ be a state on $\mathcal{A}$ such that $\varphi(a) = \|a\|$. Put $\|a\|1 - a = h^2 (h \geqslant 0, h \in \mathcal{A})$. Then, $\varphi(h^2) = 0$ and

$$\begin{aligned} & |-\varphi(D(a))-\varphi(2(\delta(h))^2)| \\ = \;& |\varphi(D(h^2-\|a\|1))-\varphi(2(\delta(h))^2)| \\ = \;& |\varphi(D(h^2))-\varphi(2(\delta(h))^2)| \\ = \;& |\varphi(d(h^2)-h^2d(1))-\varphi(2(\delta(h))^2)| \\ = \;& |\varphi(hd(h))+\varphi(d(h)h)-\varphi(h^2d(1))| \\ \leqslant \;& \varphi(h^2)^{1/2}\varphi(d(h)^2)^{1/2}+\varphi(d(h)^2)^{1/2}\varphi(h^2)^{1/2}+\varphi(h^4)^{1/2}\varphi(d(1)^2)^{1/2} \\ = \;& 0. \end{aligned}$$

Hence, $\varphi(D(a)) = -\varphi(2(\delta(h))^2)$. Suppose that $\{a_n\}$ is a sequence of self-adjoint elements in $\mathcal{A}$ such that $a_n \to 0$ and $D(a_n) \to b (\neq 0)$. Let φ_n be a state on $\mathcal{A}$ such that $\varphi_n(b + a_n) = \|b + a_n\|$, and let φ_0 be an accumulation point of $\{\varphi_n\}$ in the state space of $\mathcal{A}$. Then, we have,

$$\begin{aligned} |\varphi_{n_j}(b+a_{n_j})-\varphi_0(b)| &\leqslant |\varphi_{n_j}(b+a_{n_j})-\varphi_{n_j}(b)|+|\varphi_{n_j}(b)-\varphi_0(b)| \\ &\leqslant \|b+a_{n_j}-b\|+|\varphi_{n_j}(b)-\varphi_0(b)| \to 0 \end{aligned}$$

for some subsequence $\{n_j\}$ of $\{n\}$. Hence, $\varphi_0(b) = \|b\|$ and so

$$\varphi_0(D(b)) = -\varphi_0(2(\delta(h_b))^2),$$

where $\|b\|1 - b = h_b^2$. Also, if $\|b + a_{n_j}\|1 - (b + a_{n_j}) = h_{b+a_{n_j}}^2$ then

$$-\varphi_{n_j}(2(\delta(h_{b+a_{n_j}}))^2) = \varphi_{n_j}(D(b+a_{n_j})) = \varphi_{n_j}(D(b)+D(a_{n_j})) \to \varphi_0(D(b)+b).$$

Note that $h_{b+a_{n_j}}^2 \to h_b^2$ and $h_{b+a_{n_j}}$ and h_b are positive so that $h_{b+a_{n_j}} \to h_b$. Now, since δ is continuous, one can show that the left hand side of the above equality tends to $-\varphi_0(2(\delta(h_b))^2)$. Therefore,

$$-\varphi_0(2(\delta(h_b))^2) = \varphi_0(D(b)+b) = -\varphi_0(2(\delta(h_b))^2) + \varphi_0(b),$$

that is, $\varphi_0(b) = 0$, which is a contradiction. So the closed graph theorem shows that D is continuous and therefore d is continuous. □

Recall that if T is a linear mapping and we define T^* by $T^*(a) = T(a^*)^*$ for all $a \in \mathcal{A}$, then T^* is also linear.

Lemma 3.4. *Let $\mathcal{A} \subseteq \mathcal{A}$, $\delta : \mathcal{A} \to \mathcal{A}$ be a linear mapping and $d : \mathcal{A} \to \mathcal{A}$ be a δ-double derivation. Then, d^* is a δ^*-double derivation.*

Proof. For each $a, b \in \mathcal{A}$,

$$\begin{aligned} d^*(ab) &= d(b^*a^*)^* \\ &= [b^*d(a^*) + d(b^*)a^* + 2\delta(b^*)\delta(a^*)]^* \\ &= d^*(a)b + ad^*(b) + 2\delta^*(a)\delta^*(b). \end{aligned}$$

Hence, d^* is a δ^*-double derivation.

Proposition 3.5. *Let $\mathcal{A}$ be a C^*-algebra, $\delta : \mathcal{A} \to \mathcal{A}$ be a $*$-linear mapping and $d : \mathcal{A} \to \mathcal{A}$ be a δ-double derivation. Then, d is of the form $d_1 + d_2$, where d_1 is a $*$-δ-double derivation and d_2 is a derivation.*

Proof. We can write,

$$d = \frac{d + d^*}{2} + \frac{d - d^*}{2}.$$

Put $d_1 = \frac{d+d^*}{2}$ and $d_2 = \frac{d-d^*}{2}$. Then, d_1 is a $*$-δ-double derivation and d_2 is a derivation, since for each $a, b \in \mathcal{A}$,

$$\begin{aligned} d_1(ab) &= \frac{d + d^*}{2}(ab) \\ &= \frac{1}{2}(ad(b) + d(a)b + 2\delta(a)\delta(b) + ad^*(b) + d^*(a)b + 2\delta^*(a)\delta^*(b)) \\ &= a\frac{d + d^*}{2}(b) + \frac{d + d^*}{2}(a)b + \frac{1}{2}(4\delta(a)\delta(b)) \\ &= ad_1(b) + d_1(a)b + 2\delta(a)\delta(b) \end{aligned}$$

and

$$\begin{aligned} d_2(ab) &= \frac{d-d^*}{2}(ab) \\ &= \frac{1}{2}(ad(b)+d(a)b+2\delta(a)\delta(b)-ad^*(b)-d^*(a)b-2\delta^*(a)\delta^*(b)) \\ &= a\frac{d-d^*}{2}(b)+\frac{d-d^*}{2}(a)b. \end{aligned}$$

□

Corollary 3.6. *Let $\mathcal{A}$ be a C^*-algebra, $\delta : \mathcal{A} \to \mathcal{A}$ be a continuous $*$-linear mapping and $d : \mathcal{A} \to \mathcal{A}$ be a δ-double derivation. Then, d is continuous.*

We also have the following two results.

Theorem 3.7. *Let $\mathcal{A}$ be a C^*-algebra, $\delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ be two continuous linear mappings and $d : \mathcal{A} \to \mathcal{A}$ be a $*$-(δ, ε)-double derivation. Then, d is continuous.*

Theorem 3.8. *Let $\mathcal{A}$ be a C^*-algebra, $\delta, \varepsilon : \mathcal{A} \to \mathcal{A}$ be two continuous $*$-linear mappings and $d : \mathcal{A} \to \mathcal{A}$ be a (δ, ε)-double derivation. Then, d is continuous.*

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