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FINITELY PRESENTED GROUPS IN TOPOLOGICAL VIEW POINTS

B. MASHAYEKHY[1]* AND H. MIREBRAHIMI[2]

ABSTRACT. In this talk, using properties of fundamental groups and covering spaces of connected polyhedra and CW-complexes, we present topological proof for some famous theorems about finitely presented groups.

1. Introduction and Preliminaries

There are some famous results about subgroups of free groups, free products and finitely presented groups with complicated group theoretical proofs. A famous corollary of the Reidemeister-Schreier rewriting process [2] tells us that every subgroup of a finitely presented group with finite index is also finitely presented. In this talk, using some well-known relationship between covering spaces of connected polyhedra (simplicial complexes) and their fundamental groups, we intend to prove some results for finitely presented groups with an algebraic topological approach.

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* Speaker.

We suppose that the reader is familiar with some well-known notions such as free groups, free products and presentation in group theory and simplicial complexes (polyhedra), CW-complexes, covering spaces, and fundamental groups in algebraic topology. Let T be a connected simplicial complex, then T is called a tree if $dimT \leq 1$ and which contains no circuits. Let K be a connected simplicial complex with a maximal tree T in K. Define a group $G_{K,T}$ with the following presentation:

$$G_{K,T} = \langle (p,q) \in K \mid (p,q) \in T,\ (p,q)(q,r) = (p,r) if \{p,q,r\} \text{ is a simplex in } K \rangle.$$

The following are some facts in algebraic topology which we need in the proof of main results.

Theorem 1.1. ([3]) *(i) Let K be a connected polyhedron with a base point p, then its fundamental group $\pi_1(K,p)$ is isomorphic to $G_{K,T}$, where T is a maximal tree in K. (Note that we identify the simplicial complex K with its underlying set the polyhedron $|K|$.)*
(ii) If K is a graph i.e. a connected 1-complex, then $\pi_1(K,p)$ is a free group of rank $|\{(p,q) \in K \backslash T \mid T$ is a maximal tree in $K\}|$.
(iii) A group G is finitely presented if and only if there exists a finite connected polyhedron X with $G \cong \pi_1(X,p)$.
(iv) Let $(\tilde{X},p)$ be a covering space of X, $x_0 \in X$, and $Y = p^{-1}(x_0)$ be the fiber over x_0. Then $|Y| = [\pi_1(X,x_0) : p_*(\pi_1(\tilde{X},x_0))]$.

Theorem 1.2. ([3]) *For any group G, there exists a CW-complex $K(G)$ with*

$$\pi_1(K(G)) \cong G \text{ and } \pi_n(K(G)) = 1 \text{ for all } n \geq 2.$$

The space $K(G)$ is called an Eilenberg-Mac Lane space of G.

Remark 1.3. With respect to the way of constructing the Eilenberg-Mac Lane space [3], generators and relators of the group G are in one to one corresponding to 1-cells and 2-cells in $K(G)$.

Theorem 1.4. ([3]) *A group G is finitely presented if and only if the number of 1-cells and 2-cells in it's Eilenberg-Mac Lane space $K(G)$ is finite.*

Theorem 1.5. ([1]) *For any group G and its Eilenberg-Mac Lane space, K say, we have*

$$H_2(K) \cong M(G),$$

where $M(G)$ is the Schur multiplier of G.

Note that a space X is semilocally 1-connected if for every $x \in X$ there exists an open neighborhood U of x so that every closed path at x in U is nullhomotopic in X. For example any CW-complex, particularly any Eilenberg-Mac Lane space, is semilocally 1-connected space.

Theorem 1.6. ([3]) *If X is connected, locally path connected, and semilocally 1-connected and $G \leq \pi_1(X, x_0)$, then there exists a constructed covering space of X, $(\tilde{X}_G, p)$ such that*

$$p_*(\pi_1(\tilde{X}_G, \tilde{x}_0)) = G$$

Theorem 1.7. ([3]) *If X is a connected CW-complex and $\tilde{X}$ is a covering space of X, then $\tilde{X}$ is also a CW-complex with $dim\tilde{X} = dimX$. Moreover, if X has m k-cells, and $\tilde{X}$ is n-sheeted, then the number of k-cells in $\tilde{X}$ is exactly equal to mn.*

2. Main results

The following theorem is a result of the Reidemeister-Schreier rewriting process ([2] prop. 4.2, p. 103).

Theorem 2.1. *Every subgroup of a finitely presented group with finite index is also finitely presented.*

Proof. Let G be a finitely presented group and $H \leq G$ with finite index. By Theorem 1.1 (iii), there exists a finite connected polyhedron X with $G \cong \pi_1(X)$. Since X is connected, locally path connected and semilocally 1-connected, there exists a covering space $\tilde{X}_H$ so that $\pi_1(\tilde{X}_H) \cong H$, by Theorem 1.6. Since $[G : H] \leq \infty$, $\tilde{X}_H$ is a finite sheeted covering space of X and so by Theorem 1.7, $\tilde{X}_H$ is a finite polyhedron. Now, by Theorem 1.1 (iii), $\pi_1(\tilde{X}_H) \cong H$ is finitely presented. □

Theorem 2.2. *Any group having a finitely presented subgroup of finite index, is also finitely presented.*

Theorem 2.3. *If G is a finitely presented group, then its Schur multiplier $M(G)$ is finitely presented.*

Theorem 2.4. *Any covering group of a finite group is also a finitely presented group.*

Theorem 2.5. *The number of finitely presented groups is countable.*

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[1] DEPARTMENT OF MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P. O. BOX 1159-91775, MASHHAD, IRAN.

E-mail address: mashaf@math.um.ac.ir

[2] DEPARTMENT OF MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P. O. BOX 1159-91775, MASHHAD, IRAN.

E-mail address: hanieh_hmp@yahoo.com