QUASI $*$-METRICS AND FUZZY METRIC SPACES

Alireza Kamel Mirmostafaee

Department of Pure Mathematics
Ferdowsi University of Mashhad
P.O. Box 1159, Mashhad
Iran
Centre of Excellence in Analysis on Algebraic Structures
Ferdowsi University of Mashhad
Iran
e-mail: mirmostafaei@ferdowsi.um.ac.ir

Abstract. In this paper, we define a notion of a fuzzy metric $(X, M, *)$. Our definition enable us to define a natural topology on the space. Using Stone's metrization Theorem, we will show that the fuzzy topology on $(X, M, *)$ is metrizable. Our method will permit us to associate a system of quasi $*$-metrics to space $(X, M, *)$, which in return generate a fuzzy metric space $(X, M', *)$ in such a way that under suitable conditions $M \equiv M'$.

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1. Introduction

The celebrated paper of Zadeh [10], motivated many authors to generalize the notion of metric space to fuzzy framework. Several authors defined and studied different notions of a fuzzy metric space from different points of view (e.g. [1], [3]–[9]). In particular, George and Veeramani in [3] modified a notion of a fuzzy metric space which was introduced by Kramosil and Michalek [9]. Later, in [4], they have introduced a Hausdorff topology for this fuzzy metric. In 2000, Gregori and Romaguera [5], introduced a uniformity on the topology of this fuzzy metric to show that this topology is compatible with a metric on the space. However, we are unable to determine which metric generates the fuzzy metric topology.

In the next section of this paper, we give a new definition of a fuzzy metric space. Our definition enable us to employ Stone's Theorem, to show that the fuzzy metric space is metrizable. In Section 3, we will use of the power of our definition to define a system of quasi $*$-metrics $\{d_\alpha\}_{\alpha\in(0,1)}$ on a space X. This will let us to obtain a close relationship between a fuzzy metric space and its associated system of quasi $*$-metrics. In fact, we will show that under some restrictions, there is a one-one correspondence between fuzzy metric spaces and systems of quasi $*$-metrics.

2. Basic properties of fuzzy metric spaces

We start this section by some definitions.

Definition 2.1. Following [3], a binary operation $* : [0,1]\times[0,1] \to [0,1]$ is called a continuous t-norm if $([0,1],*)$ is an ordered abelian topological monoid with unit 1.

Example 2.2. There are numerous possible choice for a continuous t-norm $*$. For example, for each $a,b\in[0,1]$, we may define $a*b=\min\{a,b\}$, $a*b=\max\{a,b\}$, $a*b=ab$ or $a*b=a+b-ab$.

Definition 2.3. Let X be a nonempty set, by a fuzzy metric on X, we mean an ordered triple $(X,M,*)$, where $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions for all $x,y,z\in X$ and $s,t>0$.

(a) $M(x,y,t)>0$,

(b) $M(x,y,t)=1$ for all $t>0$, if and only if $x=y$,

(c) $M(x,y,t)=M(y,x,t)$,

(d) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$,

(e) $\lim\limits_{t\to\infty} M(x,y,t)=1$.

It follows from (b) and (d) that for each $x,y\in X$, $M(x,y,\cdot):(0,\infty)\to[0,1]$ is increasing. In fact, if $s<t$, then

$$M(x,y,t)\ge M(x,y,s)*M(y,y,t-s)=M(x,y,s).$$

One may regard M as a function which associates to each pair $(x,y)\in X^2$ and $t\in R^+$ the truth of the following statement:

" The distance between x and y is less then t."

Example 2.4. Let (X,d) be a metric space define

$$M(x,y,t)=\begin{cases}\dfrac{t}{d(x,y)} & \text{if } t<d(x,y)\\[2ex] 1 & \text{if } t\ge d(x,y)\end{cases}$$

It can be easily verified that $(X, M, *)$ is a fuzzy metric space, where $a * b = \min\{a, b\},\ a, b \in [0, 1]$.

The difference between our definition of a fuzzy metric space with the one given by A. George, P. Veeramani in [3] is in condition (e). In fact, by their definition, a fuzzy metric space is a 3-tuple $(X, M, *)$, where X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ which satisfy $(a) - (d)$ of the above definition and

(e$'$) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left-continuous for each $x, y \in X$.

The condition (e), in Definition 2.3, implies that for each $0 < \alpha < 1$ and $x, y \in X$, the set $\{t > 0 : M(x, y, t) \geq \alpha\}$ is a nonempty. Therefore, we can define

$$d_\alpha(x, y) = \inf\{t > 0 : M(x, y, t) \geq \alpha\}, \tag{2.1}$$

for each $0 < \alpha < 1$ and $x, y \in X$.

In the following proposition, some properties of $\{d_\alpha\}_{\alpha \in (0,1)}$ are exhibited.

Proposition 2.5. *Let $(X, M, *)$ be a fuzzy metric space. For $0 < \alpha < 1$, Then the following hold:*

(a) $d_\alpha(x, y) = 0$ *if* $x = y$.

(b) $d_\alpha(x, y) = d_\alpha(y, x)$.

(c) *If* $\alpha_1 < \alpha_2$, *then* $d_{\alpha_1}(x, y) \leq d_{\alpha_2}(x, y)$.
For all $x, y \in X$ *and* $\alpha, \alpha_1, \alpha_2 \in (0, 1)$.

Proof. (a) If $x = y$, then $M(x, y, t) = 1$ for all $t > 0$. Therefore

$$d_\alpha(x, y) = \inf\{t > 0 : M(x, y, t) \geq \alpha\} = \inf(0, \infty) = 0.$$

(b) is evident.

(c) If $\alpha_1 < \alpha_2$, then $M(x, y, t) \geq \alpha_2$ implies that $M(x, y, t) \geq \alpha_1$. Hence

$$\{t > 0 : M(x, y, t) \geq \alpha_2\} \subseteq \{t > 0 : M(x, y, t) \geq \alpha_1\}.$$

It follows that $d_{\alpha_1}(x, y) \leq d_{\alpha_2}(x, y)$. ∎

Definition 2.6. Let $(X, M, *)$ be a fuzzy metric space, $x \in X$, $\varepsilon > 0$ and $0 < \alpha < 1$. Define

$$B_\alpha(x, \varepsilon) = \{y \in X : d_\alpha(x, y) < \varepsilon\}.$$

The following result in an immediate consequence of the property (c) of Proposition 2.5.

Corollary 2.7. *If $\alpha_1, \alpha_2 \in (0, 1)$ and $\alpha_1 < \alpha_2$, then*

$$B_{\alpha_2}(x, r) \subset B_{\alpha_1}(x, r), \ \textit{for each } r > 0.$$

Theorem 2.8. *Let $(X, M, *)$ be a fuzzy metric space and $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 1$. Let τ be the topology generated by the set*

$$\{B_{\alpha_n}(x, r) : \quad x \in X,\ n = 1, 2, 3, ... \quad \textit{and} \quad r > 0\},$$

and let τ' be the topology generated by the set

$$\{B_{\alpha}(x, r) : \quad x \in X,\ 0 < \alpha < 1, ... \quad \textit{and} \quad r > 0\}.$$

Then $\tau = \tau'$.

Proof. By the definition $\tau' \subset \tau$. If $0 < \alpha < 1$, we choose some $n \in \mathbb{N}$ such that $\alpha < \alpha_n$. By Corollary 2.7, $B_{\alpha_n}(x, r) \subset B_{\alpha}(x, t)$. This proves the theorem. ∎

By the above result the following definition is well-defined.

Definition 2.9. Let $(X, M, *)$ be a fuzzy metric space. The fuzzy metric topology τ_M of the space is defined to be the topology generated by a family

$$\{B_{\alpha_n}(x, r) : \quad x \in X,\ n = 1, 2, 3, ... \text{ and } \ r > 0\},$$

where $\{\alpha_n\}$ is an arbitrary increasing sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 1$.

Remark 2.10. Let $\{\alpha_n\}$ be an increasing sequence in $(0, 1)$ such that $\alpha_n \to 1$ as $n \to \infty$. By the continuity of $*$, for every $n \in \mathbb{N}$, there is some $m > n$, such that

$$\alpha_m * \alpha_m \geq \alpha_n.$$

We use this simple observation in the following result:

Lemma 2.11. *Let $(X, M, *)$ be a fuzzy metric space and d_α be as Proposition 2.5. Let $d_n = d_{\alpha_n}$, where $\{\alpha_n\}$ is an increasing sequence in $(0, 1)$ such that $\alpha_n \to 1$ as $n \to \infty$. Then for each $n = 1, 2, 3, ...$, there is some $m > n$, such that*

$$d_n(x, y) \leq d_m(x, z) + d_m(z, y) \textit{ for each } x, y, z \in X.$$

Proof. Let x, y, z be in X and $n \in \mathbb{N}$. The above argument shows that there is some $m > n$ such that $\alpha_m * \alpha_m \geq \alpha_n$. If for some $t, s > 0$, $M(x, z, t) \geq \alpha_m$ and $M(z, y, s) \geq \alpha_m$, then

$$M(x, y, s + t) \geq \alpha_m * \alpha_m \geq \alpha_n.$$

Therefore

$$\{t > 0 : M(x, z, t) \geq \alpha_m\} + \{s > 0 : M(z, y, s) \geq \alpha_m\} \subseteq \{r > 0 : M(x, y, r) \geq \alpha_n\}.$$

Hence

$$d_n(x, y) \leq d_m(x, z) + d_m(z, y).$$ ∎

Hereafter, we fix an increasing sequence $\{\alpha_n\}$ in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 1$ and we let $d_n = d_{\alpha_n}$. In order to show that the topology τ, defined in Theorem 2.8 makes X into a Hausdorff space, we need to the following observation:

Lemma 2.12. *Let for each $n \geq 1$, $d_n(x,y) = 0$ then $x = y$.*

Proof. If $d_n(x,y) = 0$ for all $n \geq 1$, then

$$\inf\{t > 0 : M(x,y,t) \geq \alpha_n\} = 0, \ \forall n = 1,2,3,....$$

Therefore

$$M(x,y,t) \geq \alpha_n, \quad \forall n > 0 \text{ and } t > 0.$$

It follows that $M(x,y,t) = 1$ for all $t > 0$. Hence, by the definition $x = y$. ∎

Theorem 2.13. *Let $(X,M,*)$ be a fuzzy metric space and τ be its associate topology defined in Theorem 2.8, then (X,τ) is a Hausdorff space.*

Proof. Let x and y be two distinct points of X. Then by Lemma 2.12, there is some $n > 1$ such that $r_0 = d_n(x,y) > 0$. By Lemma 2.11, we can find some $m > n$, such that

$$d_n(x,y) \leq d_m(x,z) + d_m(z,y)$$

for each $z \in X$. Then

$$x \in B_m(x, r_0/3), \ y \in B_m(y, r_0/3) \text{ and } B_m(x, r_0/3) \cap B_m(y, r_0/3) = \emptyset.$$

In fact, if $z \in B_m(x, r_0/3) \cap B_m(y, r_0/3)$, then

$$r_0 = d_n(x,y) \leq d_m(x,z) + d_m(z,y) < r_0/3 + r_0/3 = 2r_0/3.$$

This contradiction proves the theorem. ∎

Definition 2.14. Let $\mathcal{U}$ be a covering of a space X and $B \subset X$, the star of B with respect to $\mathcal{U}$ is defined to be the set

$$St(B,\mathcal{U}) = \cup\{U : U \in \mathcal{U}, B \cap U \neq \emptyset\}.$$

In order to show that the topology of a fuzzy metric space is compatible with a metric, we need to the following result:

Theorem 2.15. (Stone's Theorem) *Let X be a topological space. The following statements are equivalent:*

(1) *X is metrizable.*

(2) *X is a T_0 space and there a sequence $\{\mathcal{U}_n\}$ of open coverings with the property: for each $x \in X$ and a neighborhood W of x, there is a neighborhood V of x and $n \in \mathbb{N}$ such that $St(V,\mathcal{U}_n) \subseteq W$.*

Proof. See page 196 of [2]. ∎

Now, we are ready to state the main result of this section.

Theorem 2.16. *Let $(X, M, *)$ be a fuzzy metric space. Then the topological space (X, τ_M) is metrizable.*

Proof. By Theorem 2.13, (X, τ) is Hausdorff, hence it is T_0. Let

$$\mathcal{U}_n = \left\{ B_n\left(x, \frac{1}{n}\right) : x \in X \right\},\ n = 2, 3, \ldots.$$

If $x_0 \in X$ and W is neighborhood of x_0, there is some $n \in \mathbb{N}$, such that $B_n\left(x_0, \frac{1}{n}\right) \subseteq W$. Let $k > m > 3n$ be such that

$$\alpha_m * \alpha_m \geq \alpha_n \quad \text{and} \quad \alpha_k * \alpha_k \geq \alpha_m.$$

Let $V = B_k\left(x_0; \frac{1}{k}\right)$. We will show that $St(V, \mathcal{U}_k) \subseteq W$. Let $z \in B_k\left(y, \frac{1}{k}\right)$ and $w \in B_k\left(y, \frac{1}{k}\right) \cap V$. Then

$$\begin{aligned}
d_n(z, x_0) &\leq d_m(z, y) + d_m(y, x_0) \\
&\leq \frac{1}{m} + d_m(y, x_0) \\
&\leq \frac{1}{k} + d_k(y, w) + d_k(w, x_0) \\
&\leq \frac{1}{k} + \frac{1}{k} + \frac{1}{k} = \frac{3}{k} \\
&< \frac{1}{n}.
\end{aligned}$$

Hence $B_k\left(y, \frac{1}{k}\right) \subseteq W$. It follows that $St(V, \mathcal{U}_k) \subseteq W$. By the Stone's theorem X is metrizable. ∎

Example 2.17. Let (X, d) be a metric space, define

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

where $a * b = ab$. Then, it is easy to verify that $(X, M, *)$ is a fuzzy metric space. $(M_d, \cdot)$ is called the standard fuzzy metric space induced by d. Let $\alpha_n = 1 - \frac{1}{n}$ for $n = 1, 2, \ldots$. Then

$$\begin{aligned}
d_n(x, y) &= \inf\left\{ t > 0 : \frac{t}{t + d(x, y)} \geq 1 - \frac{1}{n} \right\} \\
&= \inf\{t > 0 : t \geq (n-1) d(x, y)\} \\
&= (n-1)\ d(x, y).
\end{aligned}$$

Therefore, the topology τ, generated by $\{d_n\}$ on X is equal to τ_d, the topology of metric space d.

Remark 2.18. Let $(X, M, *)$ be a fuzzy metric space and τ_M be its associated topology. The proof of Lemma 2.11 shows that if $\alpha * \alpha \geq \alpha$ for all $\alpha \in (0, 1)$, for example when $a * b$ is equal to $\max\{a, b\}$, $\min\{a, b\}$ or $\min\{a + b - 1\}$, then each d_n satisfies the triangle inequality. It follows that in such a situation

$$\rho(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)} \qquad \forall x, y \in X$$

defines a metric on X. It can be easily verified that the metric topology τ_ρ is equal to the fuzzy metric topology τ_M on X.

Example 2.19. Let (X, d) be a metric space, define

$$M(x, y, t) = e^{-\frac{d(x,y)}{t}}, \quad \forall x, y \in X \quad \text{and} \quad t > 0.$$

Let $a * b = \min\{a, b\}$. If $M(x, y, t) \geq M(y, z, s)$ for $x, y \in X$ and $t, s > 0$, then $\frac{1}{t} d(x, y) \leq \frac{1}{s} d(y, z)$, therefore

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &\leq \frac{t}{s} d(y, z) + d(y, z) \\ &= \frac{t + s}{s} d(y, z). \end{aligned}$$

It follows that $M(x, z, t + s) \geq M(y, z, s) = \min\{M(x, y, t), M(y, z, s)\}$. By Remark 2.18, $\tau_M = \tau_\rho$.

3. Correspondence between a system of quasi ∗-metrics and a fuzzy metric on X

In this section we are going to establish one to one correspondence between fuzzy metric spaces and a system of quasi ∗-metrics. In order to show the correspondence, we need to the following definition:

Definition 3.1. Let X be a nonempty set and $* : [0, 1] \to [0, 1]$ be a continuous t-norm. Let $\{d_\alpha\}_{\alpha \in (0,1)}$ be a family of nonnegative functions on X^2, such that for each $x, y, z \in X$ and $\alpha \in (0, 1)$ the following conditions hold:

(1) $d_\alpha(x, x) = 0$;

(2) $d_\alpha(x, y) = d_\alpha(y, x)$;

(3) if $x \neq y$, then $d_\beta(x, y) > 0$ for some $\beta \in (0, 1)$;

(4) if $\beta, \gamma \in (0, 1)$ and $\beta * \gamma \geq \alpha$, then $d_\alpha(x, z) \leq d_\beta(x, y) + d_\gamma(y, x)$.

Then $\{d_\alpha\}_{\alpha\in(0,1)}$ is said to be a system of quasi $*$-metrics on X.

Theorem 3.2. *Let X be a nonempty set, $*$ be a continuous t-norm and $\{d_\alpha\}_{\alpha\in(0,1)}$ be a system of quasi $*$-metrics on X. Define*

$$M(x,y,t) = \sup\{\alpha \in (0,1) : d_\alpha(x,y) < t\}, \tag{3.1}$$

where $x, y \in X$, $t \in \mathbb{R}$ and $\alpha \in (0,1)$, then

(i) *$(X, M, *)$ is a fuzzy metric on X.*

(ii) *If $d_{(\cdot)}(x,y) : (0,1) \to \mathbb{R}$ is left continuous and*

$$d'_\alpha(x,y) = \inf\{t > 0 : M(x,y,t) \geq \alpha\}, \tag{3.2}$$

then $d_\alpha = d'_\alpha$ for each $\alpha \in (0,1)$.

Proof. To prove (i), we have to show that the conditions (a)–(e) of Definition 2.3 hold. (a) and (c) immediately follows from the definition.

If $x = y$, then $d_\alpha(x,y) = 0 < t$ for each $t > 0$ and $\alpha \in (0,1)$. Therefore $M(x,y,t) = 1$ for each $t > 0$. Conversely, suppose that $M(x,y,t) = 1$ for each $t > 0$. Then for each $\alpha \in (0,1)$ and $t > 0$, $d_\alpha(x,y) < t$. This means that $d_\alpha(x,y) = 0$ for each $\alpha \in (0,1)$, by (3) $x = y$. This proves (b).

Let $\beta = M(x,y,t)$, $\gamma = M(y,z,s)$ and $\alpha = \beta * \gamma$. Let $\beta_1 < \beta$ and $\gamma_1 < \gamma$ and $\alpha_1 = \beta_1 * \gamma_1$. It follows from the definition that

$$d_{\beta_1}(x,y) < t \quad \text{and} \quad d_{\gamma_1}(y,z) < s.$$

Therefore, by (4), we have

$$d_{\alpha_1}(x,z) \leq d_{\beta_1}(x,y) + d_{\gamma_1}(y,z) < t + s.$$

Hence $M(x,z,t+s) \geq \alpha_1$, By the continuity of $*$,

$$M(x,z,t+s) \geq \alpha = M(x,y,t) * M(y,z,s).$$

This proves that the condition (d) of Definition 2.3.

Next we will show that $\lim_{t\to\infty} M(x,y,t) = 1$. Let $\varepsilon > 0$. Take some $\alpha \in (0,1)$ such that $\alpha > 1 - \varepsilon$. Then, for each $t > d_\alpha(x,y)$, we have

$$M(x,y,t) \geq M(x,y,d_\alpha(x,y)) \geq \alpha > 1 - \varepsilon.$$

To prove (ii), we will show that

$$d_\alpha(x,y) < t \Rightarrow d'_\alpha(x,y) \leq t \quad \text{and} \quad d'_\alpha(x,y) < t \Rightarrow d_\alpha(x,y) \leq t. \tag{2.1}$$

Let $d'(x,y) < t$, then by (3.2), we have $M(x,y,t) \geq \alpha$. Hence, by (3.1), $d_{\alpha_1}(x,y) < t$ for each $\alpha_1 < \alpha$. Thanks to left continuity of $d_{(\ \ )}(x,y)$ at α, the relation $d_\alpha(x,y) \leq t$

follows. If $d_\alpha(x,y) < t$, for some $\alpha \in (0,1)$, $t > 0$ and $x, y \in X$, then by (3.1), $M(x,y,t) \geq \alpha$. Hence, by (3.2), $d'_\alpha(x,y) \leq t$. ∎

Let $(X, M, *)$ be a fuzzy metric space. By the proofs of Proposition 2.5 and Lemmas 2.11 and 2.12 the family $\{d_\alpha\}_{\alpha\in(0,1)}$, which is defined by (2.1), is a system of quasi $*$-metrics on X. The following proposition shows that there is a close relationship between M and its corresponding system of quasi $*$-metrics.

Proposition 3.3. *Let $(X, M, *)$ be a fuzzy metric space and $\{d_\alpha\}_{\alpha\in(0,1)}$ be its corresponding system of quasi $*$-metrics. Let $x, y \in X$ and $\alpha \in (0,1)$.*

(1) *If $M(x,y,\cdot) : (0,\infty) \to (0,1]$ is strictly increasing on the set $\{t > 0 : 0 < M(x,y,t) < 1\}$ and $M(x,y,s) = \alpha$ then $d_\alpha(x,y) = s$.*

(2) *If $M(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous and $d_\alpha(x,y) = s$, then $M(x,y,s) = \alpha$.*

Proof. (1) If $t_0 < s < t_1$ then $M(x,y,t_0) < M(x,y,s) < M(x,y,t_1)$. Hence

$$d_\alpha(x,y) = \inf\{t > 0 : M(x,y,t) \geq \alpha\} = \inf\{t > 0 : t \geq s\} = s.$$

(2) If $d_\alpha(x,y) = s$, it follows from the definition that

$$M(x,y,t_1) < \alpha, \quad \forall t_1 < s,$$
$$M(x,y,t_2) \geq \alpha, \quad \forall t_2 > s.$$

By the continuity of $M(x,y,\cdot)$ at s,

$$M(x,y,s) = \lim_{t\to s} M(x,y,t) = \alpha.$$

∎

Let $(X, M, *)$ be a fuzzy metric space and $\{d_\alpha\}_{\alpha\in(0,1)}$ be its corresponding system of quasi $*$-metrics on X. Let $M'(x,y,t) = \sup\{\alpha \in (0,1) : d_\alpha(x,y) < t\}$ for each $x, y \in X$ and $t > 0$. By Theorem 3.2, $(X, M', *)$ is a fuzzy metric on X. In the next result we will show that $M \equiv M'$ if for each $x, y \in X$, $M(x,y,\cdot)$ is a continuous function on $(0,\infty)$.

Theorem 3.4. *Under the above notations, we have the following.*

(1) *$M(x,y,t) \geq M'(x,y,t)$ for each $x, y \in X$ and $t > 0$.*

(2) *If $x, y \in X$ and $M(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous, then $M(x,y,t) = M'(x,y,t)$ for each $t > 0$.*

Proof. (1) Let $M'(x,y,t) > \alpha$ for $x, y \in X$, $t > 0$ and $\alpha \in (0,1)$. It follows from the definition that for each $\alpha_1 < \alpha$, $d_{\alpha_1}(x,y) < t$ and hence $M(x,y,t) \geq \alpha_1$. Since this holds for each $\alpha_1 < \alpha$, the relation $M(x,y,t) \geq \alpha$ holds. This proves (1).

(2) Let $M(x,y,t) > \alpha$ for $x, y \in X$, $t > 0$ and $\alpha \in (0,1)$. By the definition, $d_\alpha(x,y) \leq t$. If $d_\alpha(x,y) = t$, then by Proposition 3.3 (2), $M(x,y,t) = \alpha$. Therefore $d_\alpha(x,y) < t$. Hence, by the definition, $M'(x,y,t) \geq \alpha$. This together with (1) proves (2). ∎

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