Free vibration of microscaled Timoshenko beams

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In this paper, a comprehensive model is presented to investigate the influence of surface elasticity and residual surface tension on the natural frequency of flexural vibrations of microbeams in the presence of rotary inertia and shear deformation effects. An explicit solution is derived for the natural oscillations of microscaled Timoshenko beams considering surface effects. The analytical results are illustrated with numerical examples in which two types of microbeams are configured based on Euler–Bernoulli and Timoshenko beam theory considering surface elasticity and residual surface tension. The natural frequencies of vibration are calculated for selected beam length on the order of nanometer to microns and the results are compared with those corresponding to the classical beam models, emphasizing the differences occurring when the surface effects are significant. It is found that the nondimensional natural frequency of the vibration of micro and nanoscaled beams is size dependent and for limiting case in which the beam length increases, the results tends to the results obtained by classical beam models. This study might be helpful for the design of high-precision measurement devices such as chemical and biological sensors. © *2009 American Institute of Physics*. [doi:10.1063/1.3246143]

High-precision measurement techniques based on microbeams have attracted considerable interest in the last few years and due to their numerous benefits have been widely used in micro- and nanoscale technologies such as atomic force microscopy (AFM) and microelectromechanical transducers as a platform for chemical and biological sensors.[1] Several investigations concerning the classical elasticity theory for explanation of the mechanical behavior of microbeams have been reported in the literature[2] but they do not admit intrinsic size dependence in the elastic solutions of micro and nanoscaled devises.[3] In atomistic scales due to the increasing ratio between surface/interface area and volume, the importance of stress and strain effects on surface physics dominates.[4–6] Therefore, it has been a great theoretical, computational, and experimental activity that has permitted a better understanding of the stress effects on surface physics. Lagowski *et al.*[7] analyzed the natural frequency of microbeams by considering the influence of the residual surface stress on the normal mode of vibration of thin crystals. In his model, the effect of residual surface stress is represented by a compressive axial force. Gurtin *et al.*[8] modified this model by applying in addition to the compressive axial force, a distributed traction over the beam surfaces induced by the residual surface tension under bending and concluded that surface elasticity influences the natural frequency of microbeams while residual surface stress does not have any significant effect, in contrast to the results of Lagowski *et al.*[7] The surface/interface tension of fluids can be expressed by the Laplace–Young equation. Gurtin *et al.*[9] formulated a continuum model of surface elasticity in which the Laplace–Young equation which was extended to solid materials. Wang and Feng[10] estimated the natural frequencies of a microbeam in the presence of surface effects based on Euler–Bernoulli beam theory. He and Lilley[11] studied the elastic behavior of static bending of nanowires considering surface effects for different boundary conditions and compared the results of their analysis by experiment. In the present study, a more comprehensive model based on the model of Gurtin *et al.*[8] is proposed to investigate the free vibration of a microbeam in the presence of surface elasticity, residual surface tension, and rotary inertia and shear deformation.

The small flexural vibration of an elastic beam with dimensions of length l $(0 \le x \le l)$, width b $(-b/2 \le y \le b/2)$, and thickness $2h$ $(-h \le z \le h)$, as is shown in Fig. 1 is considered. The Young's modulus, shear modulus, and mass density of the beam are denoted as E, G, and ρ, respectively. In order to consider the surface effect in this study, we assume that the upper and lower surfaces of beam have surface elastic modulus E_s (Refs. 3–6 and 8) and constant residual surface tension.[10] The influence of the residual surface stress on the beam is determined by the Laplace–Young equation.[9,12] The stress jump across a surface $\langle \sigma_{ij}^{+} - \sigma_{ij}^{-} \rangle$ is related to the curvature tensor $(\kappa_{\alpha\beta})$ of the surface by[10]

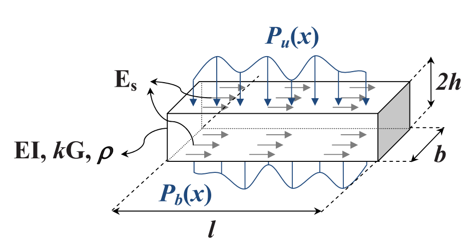

FIG. 1. (Color online) Problem geometry and properties of Timoshenko microbeam.

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$$\langle \sigma_{ij}^{+} - \sigma_{ij}^{-} \rangle n_i n_j = \sigma_{\alpha\beta}^{s} \kappa_{\alpha\beta}, \tag{1}$$

where σ_{ij}^{+} and σ_{ij}^{-} denote the stresses above and below the surface, respectively, n_i is the unit vector normal to the surface and $\sigma_{\alpha\beta}^{s}$ is the surface stresses.

The stress tensor, σ_{ij} is generally in three dimensional so the corresponding Latin indices take the values $i,j=1,2,3$ while the surface stress is two dimensional in nature and the corresponding Greek indices take the values $\alpha,\beta=1,2$. Conventional Einstein's summation rules apply unless otherwise noted.

According to Timoshenko beam theory the curvature of bending beam is approximated by[13]

$$\kappa = -\frac{\partial \phi}{\partial x}, \tag{2}$$

where ϕ is the angle of beam cross section rotation due to pure bending. By assuming the upper and lower surfaces residual surface tensions as τ_u and τ_b, respectively,[10] the Laplace–Young equation gives the distributed loading on two surfaces as

$$p_u(x) = -\tau_u b \frac{\partial \phi}{\partial x}, \tag{3}$$

$$p_b(x) = -\tau_b b \frac{\partial \phi}{\partial x}. \tag{4}$$

The energy method was used to calculate the differential equation of the Timoshenko beam due to the surface effects. The total potential energy of the system contains the two following parts: (i) the elastic strain energy of the bulk (including shear deformation) and (ii) the elastic strain energy of the surfaces. Furthermore, the kinematic energy of the system contains the rotary inertia effect and the residual surface tension acting as an external distributed load on the beam whose work should be calculated. By implementation of the generalized Hamilton's principle we obtain the following partial differential equations for vibration of a Timoshenko type beam due to effects of surface elasticity and residual surface tension:

$$(EI + 2bh^2 E_s)\frac{\partial^2 \phi}{\partial x^2} - kGA\phi + kGA\frac{\partial w}{\partial x} = \rho I \frac{\partial^2 \phi}{\partial t^2}, \tag{5}$$

$$kGA\frac{\partial^2 w}{\partial x^2} + [(\tau_u + \tau_b)b - kGA]\frac{\partial \phi}{\partial x} = \rho A \frac{\partial^2 w}{\partial t^2}, \tag{6}$$

where $A=2bh$ is the cross-section area, $I=2bh^3/3$ the inertia moment of it, and k known as the shear deformation factor. Differentiating Eq. (5) with respect to x and substitution $(\partial\phi/\partial x)$ from Eq. (6) yields the desired equation

$$(\alpha^2 E + \alpha_s^2 E_s)\frac{\partial^4 w}{\partial x^4} - \frac{\tau b}{\rho A}\frac{\partial^2 w}{\partial x^2} - \left(r^2 + \frac{E}{kG}r^2 + \frac{E_s}{kG}r_s^2\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^2 w}{\partial t^2} + \frac{\rho r^2}{kG}\frac{\partial^4 w}{\partial t^4} = 0, \tag{7}$$

where $\alpha^2=I/\rho A$, $\alpha_s^2=2bh^2/\rho A$, $r^2=I/A$, $r_s^2=2bh^2/A$, and $\tau=\tau_u+\tau_b$.

The natural frequencies of vibration of the microbeam expressed by Eq. (7) can be calculated by assuming a harmonic time variation and solving Eq. (7) while satisfying the specific boundary conditions of the beam. In the current article, this approach implemented for a simply supported microbeam in the following application.

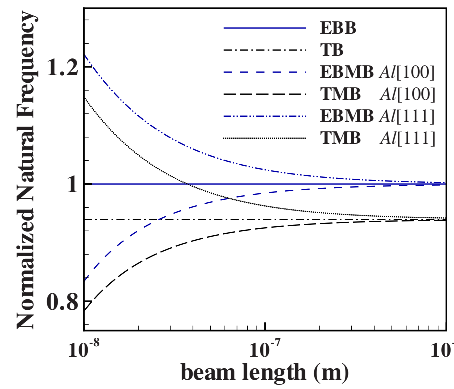

FIG. 2. (Color online) Influence of surface effects, rotary inertia, and shear deformation on the nondimensional fundamental natural frequency of the microbeam for $2h=0.2l$ and $k=5/6$.

The solution of Eq. (7), which also satisfies the simply supported boundary conditions, is assumed as

$$w(x,t) = C \sin\frac{n\pi x}{l}\cos \omega_n t, \tag{8}$$

where C is a constant and ω_n is the nth natural frequency of vibration. Substitution of Eq. (8) into Eq. (7) gives the frequency equation

$$\frac{\rho r^2}{kG}\omega_n^4 - \left(1 + \frac{n^2\pi^2 r^2}{l^2} + \frac{E}{kG}\frac{n^2\pi^2 r^2}{l^2} + \frac{E_s}{kG}\frac{n^2\pi^2 r_s^2}{l^2}\right)\omega_n^2 + E\frac{n^4\pi^4\alpha^2}{l^4} + E_s\frac{n^4\pi^4\alpha_s^2}{l^4} + \frac{\tau b}{\rho A}\frac{n^2\pi^2}{l^2} = 0. \tag{9}$$

Equation (9) is a quadratic equation in ω_n^2 and gives two values of ω_n^2 for any value of n. The smaller value corresponds to the bending deformation mode, and the larger one corresponds to the shear deformation mode.[13]

In order to illustrate the nature and general behavior of the solution, we present the numerical examples (Figs. 2–4). The embedded atom method was used by Miller and Shenoy[4] and Shenoy[14] to determine the surface elastic constants. Their results indicated that the surface elastic constants depend on the material type and the surface crystal orientation. For example, for an anodic alumina (Young's modulus $E=70$ GPa, Poisson's ratio $\nu=0.3$ and $\rho=2700$ kg/m^3) with crystallographic direction of [100] the related properties of the surface are $E_s=-7.9253$ N/m and $\tau=0.5689$ N/m; while for crystallographic direction [111], $E_s=5.1882$ N/m and $\tau=0.9108$ N/m.[15] In presenting the numerical examples, the above parameters are taken into account for material properties and related surface parameters of the microbeams. The solutions based on classical Euler–Bernoulli and Timoshenko beam theory are denoted by EBB and TB, respectively. The other results obtained for microbeams including the surface effects are denoted by EBMB for Euler–Bernoulli beam theory and TMB for Timoshenko beam theory. The natural frequencies are normalized to $(1/n^2\pi^2)\sqrt{\rho A l^4/EI}$ in the following examples.

Figure 2 illustrates the size dependence in the nondimensional natural frequency of EBMB and TMB microbeams in comparison to classical solutions of EBB and TB

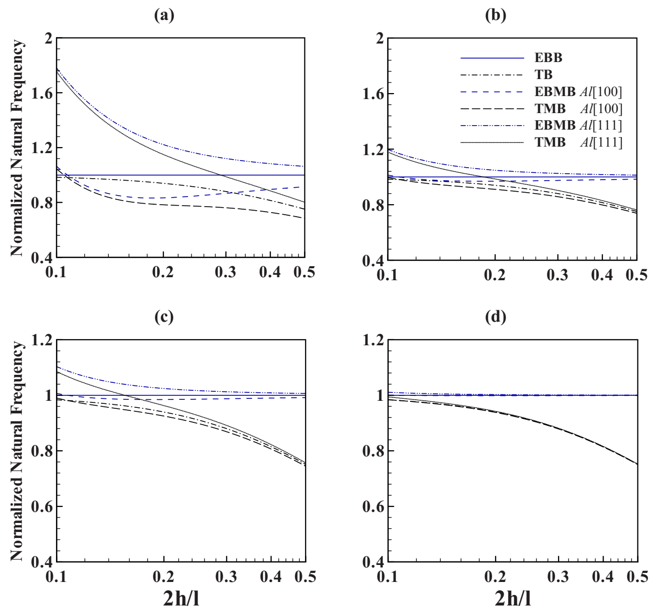

FIG. 3. (Color online) Effect of beam thickness on the nondimensional fundamental natural frequency of the microbeams for $k=5/6$, and different beam length: (a) $l=10$ nm, (b) $l=50$ nm, (b) $l=100$ nm, and (d) $l=1$ μm.

beams. It can be seen that the natural frequency of TMB microbeams are smaller than those calculated for EBMB ones. It is found that the nondimensional natural frequency of the vibration of EBB and TB beams is independent of the beam length while for EBMB and TMB the situation is different. For beam length on the order of nanometer to microns, the difference between natural frequencies is apparent and by increasing the length of the microbeam, the results tend to EBB and TB, respectively.

Figure 3 shows the variation of the first nondimensional natural frequency of the microbeam as a function of beam thickness in comparison to classical EBB and TB beams for selected beam lengths. In this figure similar to EBMB, a positive surface elastic constant will increase the natural frequency, while a negative value will decrease the natural frequency. Also for EBMB and TMB microbeams when the beam length increases from nanometers to microns and larger values, the surface effects disappear and the results converge into natural frequencies of classical EBB and TB beams, respectively.

Figure 4 displays the nondimensional natural frequencies of a microbeam for different modes of vibration. Furthermore, the obtained results are compared to natural frequencies of classical EBB and TB beams. It can be seen that the corresponding value of natural frequencies of TMB are smaller than those obtained for EBMB and the deference between these values increases as the mode number increases. It is found that the influence of the related surface parameters is considerably higher in lower modes. In the case of EBMB, the nondimensional natural frequencies of the vibration reach to a constant value as the mode number increases while for TMB the natural frequencies decreases with the increase in mode number. On the other hand, when the beam length increases from nanometers to microns and larger values, the surface effects disappear and the results converge into two distinct branches which are very close to natural frequencies of classical EBB and TB.

In conclusion, our analysis predicts the size dependence in free vibration analysis of microscaled Timoshenko beams. The outcome of the theoretical analysis shows that in addition to surface effects, rotary inertia and shear deformation can affect significantly on the natural frequencies of microbeams. The comprehensive model presented in this paper is crucial for design of nanoscaled chemical and biological measurement devices.

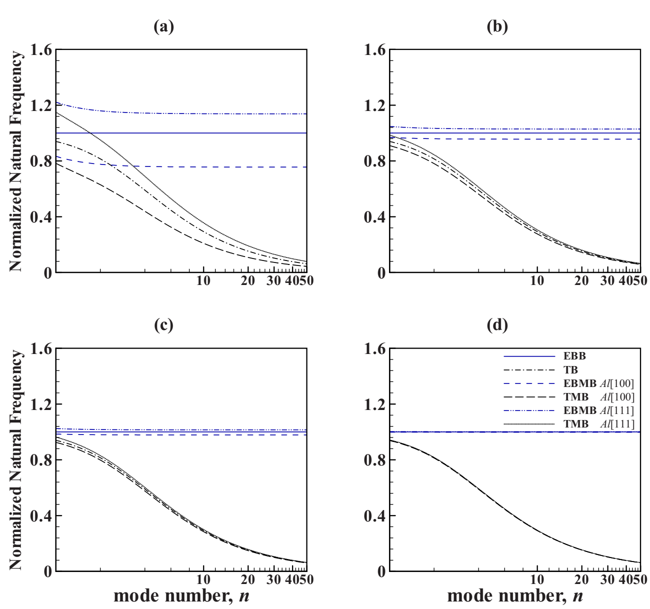

FIG. 4. (Color online) Effect of mode number on the nondimensional natural frequencies of the microbeams for $2h=0.2l$, $k=5/6$ and different beam length: (a) $l=10$ nm, (b) $l=50$ nm, (b) $l=100$ nm, and (d) $l=1$ μm.

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