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C^1 ROBUSTLY MINIMAL ITERATED FUNCTION SYSTEMS

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We construct iterated function systems on compact manifolds that are C^1 robustly minimal. On the m-dimensional torus and on two-dimensional compact manifolds, examples are provided of C^1 robustly minimal iterated function systems that are generated by just two diffeomorphisms.

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1. Introduction

Our motivation for this paper comes from a result contained in [3] by Gorodetskiĭ and Il'yashenko on iterated function systems on the circle. They provide an example of an iterated function system generated by two circle diffeomorphisms, that is robustly minimal in the C^1 topology. The example consists of an irrational rigid rotation and a diffeomorphism with an attracting and a repelling fixed point; we refer to [6, Proposition 12] for details of the construction.

We generalize this example to iterated function systems on m-dimensional compact Riemannian manifolds M by constructing examples of C^1 robustly minimal

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iterated function systems on M. On the m-dimensional torus $\mathbb{T}^m = (\mathbb{R}/\mathbb{Z})^m$ and on two-dimensional compact manifolds, i.e. compact surfaces, robust minimal iterated function systems with two generators exist. A somewhat related problem on minimality of an iterated function system generated by a generic pair of area preserving diffeomorphisms was recently raised in [7].

We begin to introduce definitions and notations of iterated function systems, and then formulate our main results. Consider a collection of diffeomorphisms $\mathcal{L} = \{g_1, \ldots, g_n\}$ on M. The iterated function system $\mathcal{G}(M; g_1, \ldots, g_n)$ on M generated by $\mathcal{L}$ is given by iterates $g_{i_1} \circ \cdots \circ g_{i_k}$ with $i_j \in 1, \ldots, n$. As is well known, iterated function systems are a popular way to generate and explore a variety of fractals [1, 2]. Consider the space $\mathrm{Diff}^1(M)$ of C^1 diffeomorphisms of M, endowed with the C^1 topology. Recall that a map $f : M \to M$ is minimal if each closed subset $X \subset M$ such that $f(X) \subset X$ is empty or coincides with M. An iterated function system $\mathcal{G}(M; g_1, \ldots, g_s)$ on M is minimal if each closed subset $A \subset M$ such that $g_i(A) \subset (A)$ for all i is empty or coincides with M. Equivalently, for a minimal iterated function system $\mathcal{G}(M; g_1, \ldots, g_s)$, for any $x \in M$ the collection of iterates $g_{i_1} \circ \cdots \circ g_{i_k}(x)$, $i_j \geq 0$, is dense in M.

Theorem 1.1. *Let M be a compact connected m-dimensional manifold. Then there exist diffeomorphisms $T_1, \ldots, T_{m+3}$ on M and a neighborhood*

$$U \subset \underbrace{\mathrm{Diff}^1(M) \times \cdots \times \mathrm{Diff}^1(M)}_{m+3 \text{ times}}$$

of $(T_1, \ldots, T_{m+3})$ such that each element in U forms a minimal iterated function system on M.

This result raises the question of the minimal number of generators of C^1 robustly minimal iterated function systems. The following two results provide answers in two cases, iterated function systems on tori and compact surfaces.

Theorem 1.2. *There exist two diffeomorphisms T_1, T_2 on the m-dimensional torus $\mathbb{T}^m$ and a neighborhood $U \subset \mathrm{Diff}^1(M) \times \mathrm{Diff}^1(M)$ of (T_1, T_2) such that each element in U forms a minimal iterated function system on $\mathbb{T}^m$.*

Theorem 1.3. *Let M be a compact connected surface. There exist two diffeomorphisms T_1, T_2 on M and a neighborhood $U \subset \mathrm{Diff}^1(M) \times \mathrm{Diff}^1(M)$ of (T_1, T_2) such that each element in U forms a minimal iterated function system on M.*

These theorems are proved in the following section.

2. Robust Minimal Iterated Function Systems

For $x \in M$, define $\Gamma(x) \subset \mathrm{Diff}^1(M)$ by

$$\Gamma(x) = \left\{ g \in \mathrm{Diff}^1(M) \,\middle|\, 1 < \|Dg^{-1}(g(x))\| < 2 \text{ and } \frac{1}{2} < \|Dg(x)\| < 1 \right\}.$$

Given a small open neighborhood V of a point $a \in M$, put

$$C_V = \{g \in \mathrm{Diff}^1(M) \mid g(\bar{V}) \subset V \text{ and } \forall\, x \in \bar{V}, g \in \Gamma(x)\}.$$

Note that in the C^1-topology, C_V is open and the map $\alpha : C_V \to V$ that takes each map to its fixed point in V is continuous.

We start with some lemmas.

Lemma 2.1. *For $\mathcal{L} = \{g_1, \ldots, g_s\}$ with $g_i \in C_V$, $i = 1, \ldots, s$, there exists a unique non-empty compact set Δ such that the iterated function system $\mathcal{G}(\triangle; g_1, \ldots, g_s)$ is minimal.*

Proof. Take $\mathcal{L}^0(\bar{V}) = \bar{V}$, $\mathcal{L}^1(\bar{V}) = \mathcal{L}(\bar{V}) = \bigcup_{i=1}^{n} g_i(\bar{V})$, $\mathcal{L}^p(\bar{V}) = \mathcal{L}(\mathcal{L}^{p-1}(\bar{V}))$ for $p > 1$. Since $\mathcal{L}(\bar{V}) \subset V$,

$$\bar{V} \supset \mathcal{L}(\bar{V}) \supset \mathcal{L}^2(\bar{V}) \supset \cdots \supset \mathcal{L}^p(\bar{V}) \supset \cdots$$

Now $\triangle = \lim_{p\to\infty} \mathcal{L}^p(\bar{V})$ is a nonempty compact set that is invariant for $\mathcal{L}$. Since g_i are contractions on $\bar{V}$, $\triangle$ is the unique compact set that is invariant for $\mathcal{L}$ [4]. Thus the iterated function system $\mathcal{G}(\triangle; g_1, \ldots, g_s)$ is minimal. □

An ordered set of points $\{p_1, \ldots, p_{m+1}\} \subset \mathbb{R}^m$ is called affine independent if $\{\overrightarrow{p_1p_2}, \overrightarrow{p_1p_3}, \ldots, \overrightarrow{p_1p_{m+1}}\}$ is linearly independent.

Lemma 2.2. *In Lemma* 2.1, *it is possible to choose $\mathcal{L} = \{g_1, \ldots, g_{m+1}\}$, $g_i \in C_V$ for $1 \le i \le m+1$, so that the interior of $\triangle$ is nonempty.*

Proof. In coordinates we may assume that $V \subset \mathbb{R}^m$. Choose $\{g_1, \ldots, g_{m+1}\} \subset C_V$ such that the subset $\{\alpha(g_1), \ldots, \alpha(g_{m+1})\}$ is affine independent. Moreover, choose g_i, $i = 1, \ldots, m+1$, so that $Dg_i(\alpha(g_i))$ is a multiple of the identity. Take a linear system $\tilde{\mathcal{L}} = \{k_1, \ldots, k_{m+1}\}$, where $k_i(x) = Dg_i(\alpha(g_i))(x - \alpha(g_i)) + \alpha(g_i)$, $i = 1, \ldots, m+1$. By shrinking V, if necessary, k_i is arbitrary close to g_i on $\bar{V}$. It is clear that the set $\tilde{\triangle} = \mathrm{conv}\{\alpha(g_1), \ldots, \alpha(g_{m+1})\}$ is an invariant set for $\tilde{\mathcal{L}}$, where $\mathrm{conv}\{a_1, \ldots, a_{m+1}\}$ is the convex hull spanned by $\{a_1, \ldots, a_{m+1}\}$.

Take $\alpha'_i \subset \mathrm{int}(\tilde{\triangle})$ close to $\alpha(g_i)$, $i = 1, \ldots, m+1$. Let $\triangle^2 = \mathrm{conv}\{\alpha'_1, \ldots, \alpha'_{m+1}\}$. Then

$$\triangle^2 \subset \tilde{\mathcal{L}}(\triangle^2) \subset \cdots \subset \tilde{\mathcal{L}}^n(\triangle^2) \subset \cdots$$

which implies that $\lim_{n\to\infty} \tilde{\mathcal{L}}^n(\triangle^2) = \bigcup_{n\ge 0} \tilde{\mathcal{L}}^n(\triangle^2) = \mathrm{int}\tilde{\triangle}$. Since g_i is close to k_i on $\bar{V}$,

$$\triangle^2 \subset \mathcal{L}(\triangle^2) \subset \cdots \subset \mathcal{L}^n(\triangle^2) \subset \cdots \tag{1}$$

and hence $\mathrm{int}\triangle \supset \bigcup_{i\ge 0} \mathcal{L}^n(\triangle^2)$. □

The proof of Lemma 2.2 gives more than its statement as it includes arguments for C^1 robust occurrence of invariant sets with nonempty interior.

Corollary 2.3. *Let* $\{g_1,\dots,g_{m+1}\}$, $g_i \in C_V$ *for* $1 \le i \le m+1$, *be such that* $\{\alpha(g_1),\dots,\alpha(g_{m+1})\}$ *is affine independent (assuming, as in the proof of Lemma* 2.2, *that* $V \subset \mathbb{R}^m$). *Then there exists a neighborhood* $W \subset \underbrace{\mathrm{Diff}^1(M)\times\cdots\times\mathrm{Diff}^1(M)}_{m+1 \text{ times}}$ *of* $(g_1,\dots,g_{m+1})$ *such that each element* $\mathcal{F}=(f_1,\dots,f_{m+1})$ *in this neighborhood admits an invariant set with non-empty interior.*

The above lemmas are ingredients in the proof of Theorem 1.1.

Proof of Theorem 1.1. We prove the theorem by establishing the following: there exist an open neighborhood V of a point $q \in M$, an iterated function system $\mathcal{L}=\{g_1,\dots,g_{m+1}\}$ with $g_i \in C_V$ for $1 \le i \le m+1$, a diffeomorphism T on M, and a neighborhood $U \subset (\mathrm{Diff}^1(M))^{m+3}$ of $(T, T^{-1}, g_1,\dots,g_{m+1})$ such that each element in U forms a minimal system on M.

Take a gradient Morse–Smale vector field on M with a unique hyperbolic repelling equilibrium q (see e.g. [8, Theorem 3.35] for the existence of Morse functions with unique extrema) and let T be its time-1 map. Let V be a small open neighborhood of q. By following the argument in the proof of Lemma 2.2 for $\mathcal{L}=\{g_1,\dots,g_{m+1}\}$, we can choose $\alpha'_i \in \mathrm{int}(\triangle)$ sufficient close to $\alpha(g_i)$ for $i=1,\dots,m+1$ such that $\triangle^2=\mathrm{conv}\{\alpha'_1,\dots,\alpha'_{m+1}\} \subset \mathrm{int}\triangle$. We may assume that q lies in the interior of Δ^2. The unstable manifold of q for T lies dense in M. Iterates of Δ under T therefore cover a dense subset of M. Finally, choose $g_1,\dots,g_{m+1}$ so that each of the finitely many critical points of T is mapped into the unstable manifold of q for T by at least one of the maps g_i. It is now easily seen that the iterated function system $\{T, T^{-1}, g_1,\dots,g_{m+1}\}$ is minimal.

For $\tilde{g}_i$ C^1-close to g_i, the system $\{\tilde{g}_1,\dots,\tilde{g}_{m+1}\}$ is minimal on a compact set $\tilde{\Delta}$ containing Δ^2. A diffeomorphism $\tilde{T}$ that is C^1-close to T has its fixed points near those of T, in particular a unique hyperbolic repelling equilibrium $\tilde{q}$ inside Δ^2 with dense unstable manifold [9]. It follows that a system $\{\tilde{T}, \tilde{T}^{-1}, \tilde{g}_1,\dots,\tilde{g}_{m+1}\}$ whose generators are sufficiently C^1-close to those of $\{T, T^{-1}, g_1,\dots,g_{m+1}\}$ is minimal. □

We finish with the examples of robust minimal systems on tori and compact surfaces. The proofs use arguments similar to the ones above.

Proof of Theorem 1.2. Let T_1 be a C^1 diffeomorphism of $\mathbb{T}^m$ possessing an attracting fixed point $a=(a_1,\dots,a_m)$, so that $T_1 \in \Gamma(a)$ and $DT_1(a)$ is a diagonal matrix. Let T_2 be a minimal translation on $\mathbb{T}^m$, see e.g. [5, Sec. 1.4].

Since T_2 is a minimal translation and C_V is open, there exist $n_i \in \mathbb{N}$ such that

$$g_i = T_2^{n_i} T_1 \in C(V)$$

for $i = 1, \dots, m+1$ and the set $\{\alpha(g_1), \dots, \alpha(g_{m+1})\}$ is affine independent. We can apply the arguments in the proof of Lemma 2.2 for $\mathcal{L} = \{g_1, \dots, g_{m+1}\}$; let Δ^2 be as in that proof so that (1) applies.

Iterates of $\triangle^2$ under T_2 cover $\mathbb{T}^m$; thus there exists a finite subcover Ω^+. Moreover, since T_2^{-1} is also minimal, it follows that there exists a finite subcover Ω^- of $\mathbb{T}^m$ by the images of $\triangle^2$ under the iterates of T_2^{-1}. The same applies when T_2 is replaced by $\tilde{T}_2$ from a small neighborhood of T_2 in $\mathrm{Diff}^1(\mathbb{T}^m)$.

By Corollary 2.3, there exists a neighborhood W of $(g_1, \dots, g_{m+1})$ such that for each element $(f_1, \dots, f_{m+1})$ in this neighborhood, the iterated function system $\{f_1, \dots, f_{m+1}\}$ has a unique compact set $\triangle_{\mathcal{F}}$ with non-empty interior on which it is minimal. The interior of $\triangle_{\mathcal{F}}$ contains Δ^2.

We conclude that there is a small neighborhood U of (T_1, T_2) so that for each $(\tilde{T}_1, \tilde{T}_2)$ in U, $\{f_1, \dots, f_{m+1}\}$ with $f_i = \tilde{T}_2^{n_i} \tilde{T}_1$ acts minimally on a set that contains Δ^2, and forward and backward iterates under $\tilde{T}_2$ of Δ^2 cover $\mathbb{T}^m$. This implies that $(\mathbb{T}^m, T_1, T_2)$ is C^1 robustly minimal. □

Proof of Theorem 1.3. Let T_1 be the time-1 map of a gradient Morse–Smale flow on M, with a unique attracting fixed point q_1 and a unique repelling fixed point p_1. Let T_2 likewise be the time-1 map of a gradient Morse–Smale flow on M, with a unique attracting fixed point q_2 and a unique repelling fixed point p_2, with q_2 close to p_1 and p_2 close to q_1. Further, choose T_1, T_2 so that T_2 maps fixed points of T_1 into the stable manifold of q_1 (which lies dense in M).

Take T_1 with $DT_1(q_1)$ close to identity and with complex conjugate eigenvalues. Take T_1 and T_2 so that further $T_2 T_1^k$ for some $k \in \mathbb{N}$ possesses a hyperbolic attracting fixed point q_{12} near q_1, with $DT_2 T_1^k(q_{12})$ close to identity and with complex conjugate eigenvalues. Moreover, we can ensure that T_1 and $T_2 T_1^k$ are linear in coordinates near $\{q_1\} \cup \{q_{12}\}$, and time-1 maps of flows φ_1^t, φ_{12}^t.

We claim that $\mathcal{L} = \{T_1, T_{12}\}$ is minimal on a unique compact set Δ with a nonempty interior that contains q_1, q_{12}. To see this, write l for the closed line piece connecting q_1 and q_{12} and consider the set U which is the union of $\cup_{t \geq 0} \varphi_1^t(l)$ and $\cup_{t \geq 0} \varphi_{12}^t(l)$. Since q_1 and q_{12} are attracting fixed points for T_1 and $T_1 T_2$, U is a closed set that contains q_1 and q_{12} in its interior. Since $DT_1(q_1)$ and $DT_1 T_2(q_{12})$ are near the identity, one has $\mathcal{L}(U) \supset U$. Hence $U \subset \Delta$. Noting that U contains a slightly longer linepiece that extends l, one sees that $U \subset \operatorname{int} \mathcal{L}^n(U)$ for some $n \in \mathbb{N}$. Then, for $\tilde{T}_1$, $\tilde{T}_2$ C^1-close to T_1, T_2, also $U \subset \operatorname{int} \tilde{\mathcal{L}}^n(U)$ for $\tilde{\mathcal{L}} = \{\tilde{T}_1, \tilde{T}_2\}$. Thus $\tilde{\mathcal{L}}$ is minimal on a compact set that contains U in its interior.

Finally, choose T_1, T_2 so that the repelling fixed point p_2 of T_2 will be contained in U. It is easily checked that this can be done: in complex coordinates near $\{q_1\} \cup \{q_{12}\}$ one may take $T_1(z) = az$, $T_2(z) = bz + 1$ and $T_2 T_1^k(z) = ba^k z + 1$, and by varying a, b, k while keeping ba^k fixed one may assume that the fixed point of T_2 is near 0 and hence lies in U. The properties of T_1, T_2 now imply that $\mathcal{G}(M; T_1, T_2)$ is C^1 robustly minimal, compare the proof of Theorem 1.1. □

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References

1. M. F. Barnsley, *Fractals Everywhere*, 2nd edn. (Academic Press, 1993).
2. K. Falconer, *Techniques in Fractal Geometry* (John Wiley & Sons, 1997).
3. A. S. Gorodetskiĭ and Yu. S. Il′yashenko, Certain properties of skew products over a horseshoe and a solenoid, *Proc. Steklov Inst. Math.* **231** (2000) 90–112.
4. J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981) 713–747.
5. A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications, Vol. 54 (Cambridge Univ. Press, 1995).
6. V. A. Kleptsyn and M. B. Nalskii, Contraction of orbits in random dynamical systems on the circle, *Funct. Anal. Appl.* **38** (2004) 267–282.
7. A. Koropecki and M. Nassiri, Transitivity of generic semigroups of area-preserving surface diffeomorphisms, to appear in *Math. Z.*
8. Y. Matsumoto, *An Introduction to Morse Theory*, Translations of Mathematical Monographs, Vol. 208 (Amer. Math. Soc., 2002).
9. J. Palis, On Morse–Smale dynamical systems, *Topology* **8** (1968) 385–404.