Bulletin of the Iranian Mathematical Society Vol. 35 No. 2 (2009), pp 67-74.

AN OPERATOR EXTENSION OF BOHR'S INEQUALITY

M.S. MOSLEHIAN, J.E. PEČARIĆ AND I. PERIĆ *

Communicated by Heydar Radjavi

ABSTRACT. We establish an operator extension of the following generalization of Bohr's inequality, due to M.P. Vasić and D.J. Kečkić:

$$\left|\sum_{i=1}^{n} z_i\right|^r \leq \left(\sum_{i=1}^{n} \alpha_i^{1/(1-r)}\right)^{r-1} \sum_{i=1}^{n} \alpha_i |z_i|^r$$

$$(r > 1, z_i \in \mathbb{C}, \alpha_i > 0, 1 \leq i \leq n)\,.$$

We also present some norm inequalities related to our noncommutative generalization of Bohr's inequality.

1. **Introduction**

Let $\mathfrak{A}$ be a C^*-algebra of Hilbert space operators and let T be a locally compact Hausdorff space. A field $(A_t)_{t\in T}$ of operators in $\mathfrak{A}$ is called a continuous field of operators if the function $t \mapsto A_t$ is norm continuous on T. If μ is a Radon measure on T and the function $t \mapsto \|A_t\|$ is integrable, then one can form the Bochner integral $\int_T A_t \mathrm{d}\mu(t)$, which is the unique element in $\mathfrak{A}$ such that

$$\varphi\left(\int_T A_t \mathrm{d}\mu(t)\right) = \int_T \varphi(A_t)\mathrm{d}\mu(t)$$

MSC(2000): Primary: 47A63; Secondary:47B10, 47A30, 47B15, 15A60.
Keywords: Bohr's inequality, operator norm, operator inequality, positive operator, Hilbert space.
Received: 20 September 2008, Accepted: 03 November 2008.
∗Corresponding author

for every linear functional φ in the norm dual $\mathfrak{A}^*$ of $\mathfrak{A}$; cf. [3, Section 4.1].
Furthermore, a field $(\varphi_t)_{t\in T}$ of positive linear mappings $\varphi : \mathfrak{A} \to \mathfrak{B}$ between C^*-algebras of operators is called continuous if the function $t \mapsto \varphi_t(A)$ is continuous for every $A \in \mathfrak{A}$. If the C^*-algebras include the identity operators, denoted by the same I, and the field $t \mapsto \varphi_t(I)$ is integrable with integral I, then we say that $(\varphi_t)_{t\in T}$ is unital.

The classical Bohr's inequality states that for any $z, w \in \mathbb{C}$ and any positive real numbers r, s with $\frac{1}{r} + \frac{1}{s} = 1$,

$$|z + w|^2 \leq r|z|^2 + s|w|^2.$$

This inequality admits the operator extension,

$$|A + B|^2 \leq r|A|^2 + s|B|^2$$

for operators A, B in the algebra $\mathbb{B}(\mathrm{H})$ of all bounded linear operators on a complex Hilbert space H (to see this, use the Cauchy–Schwarz inequality and the fact that the operator C is positive if and only if $\langle Cx, x\rangle \geq 0$).
Over the years, interesting generalizations of this inequality have been obtained in various settings; cf. [2, 6, 7, 8, 9, 11]. There is one of special interest given by M.P. Vasić and D.J. Kečkić [10]:
If $z_1, \cdots, z_n$ are complex numbers, $r > 1$ and $\alpha_i > 0\, (i = 1, 2, \cdots, n)$, then

$$\left|\sum_{i=1}^{n} z_i\right|^r \leq \left(\sum_{i=1}^{n} \alpha_i^{1/(1-r)}\right)^{r-1} \sum_{i=1}^{n} \alpha_i |z_i|^r . \tag{1.1}$$

This is indeed an immediate consequence of the Hölder inequality. Here, we establish the operator version of inequality (1.1) and apply the obtained operator inequalities to obtain some norm inequalities related to our operator extension of Bohr's inequality.

2. **Main results**

Recall that a continuous real function f defined on a real interval J of any type is said to be operator convex if $f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$ holds for all $\lambda \in [0, 1]$ and all self-adjoint operators A, B acting on a Hilbert space with spectra in J. For instance, $f(x) = x^r$, where, $1 \leq r \leq 2$ is operator convex on $[0, \infty)$; see [1, p. 123]. In [5], the authors gave a general formulation of Jensen's inequality for unital fields

of positive linear mappings in which they dealt with operator convex functions.
We need the main result [5, Theorem 2.1]. We state it for the sake of convenience.

Theorem 2.1. *Let f be an operator convex function on an interval J and let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^*-algebras. If $(\varphi_t)_{t\in T}$ is a unital field of positive linear mappings $\varphi_t : \mathfrak{A} \to \mathfrak{B}$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then the inequality*

$$f\left(\int_T \varphi_t(A_t)d\mu(t)\right) \le \int_T \varphi_t(f(A_t))d\mu(t)\,.$$

holds for every bounded continuous field $(A_t)_{t\in T}$ of self-adjoint elements of $\mathfrak{A}$ with spectra contained in J.

Utilizing the theorem above we prove our main result.

Theorem 2.2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras of operators containing I, T be a locally compact Hausdorff space equipped with a bounded Radon measure μ, (α_t) a bounded continuous nonnegative function such that $(\alpha_t) \in L^{\frac{1}{1-r}}(T,\mu)$ and $1 < r \le 2$. Also, let $(A_t)_{t\in T}$ be a bounded continuous field of positive elements in $\mathfrak{A}$ and $(\varphi_t)_{t\in T}$ be a field of positive linear mappings $\varphi_t : \mathfrak{A} \to \mathfrak{B}$ defined on T satisfying*

$$\int_T \alpha_t^{1/(1-r)}\varphi_t(I)d\mu(t) \le \int_T \alpha_t^{1/(1-r)}d\mu(t)I\,.$$

Then,

(2.1)
$$\left(\int_T \varphi_t(A_t)d\mu(t)\right)^r \le \left(\int_T \alpha_t^{1/(1-r)}d\mu(t)\right)^{r-1}\int_T \alpha_t\varphi_t(A_t^r)d\mu(t)\,.$$

Proof. Let ∞ be an object not belonging to T. Consider $T_\infty := T \cup \{\infty\}$ as a locally compact topological space by equipping $\{\infty\}$ with the discrete topology and extend μ on T_∞ by $\mu(\{\infty\}) = 1$. Set $f(x) = x^r$ $(x \in [0,\infty))$, $\tilde{\varphi}_t := \frac{P_t}{Q}\varphi_t$ $(t \in T_\infty)$, where $P_t := \alpha_t^{1/(1-r)}$, $P_\infty := 1$, $Q := \int_T \alpha_t^{1/(1-r)}d\mu(t)$ and

$$\varphi_\infty(A) := \langle Ae, e\rangle\left(\int_T P_t d\mu(t)I - \int_T P_t\varphi_t(I)d\mu(t)\right),$$

in which A belongs to the C^*-algebra $\mathfrak{A}$ acting on a Hilbert space H and $e \in$ H is a fixed unit vector. Then,

$$\begin{aligned}\int_{T_\infty} \tilde{\varphi}_t(I)d\mu(t) &= \frac{1}{Q}\int_T P_t\varphi_t(I)d\mu(t) + \tilde{\varphi}_\infty(I) \\ &= \frac{1}{Q}\int_T P_t\varphi_t(I)d\mu(t) \\ &\quad +\frac{1}{Q}\left(\int_T P_t d\mu(t)I - \int_T P_t\varphi_t(I)d\mu(t)\right) \\ &= I\,.\end{aligned}$$

It follows from Theorem 2.1 that

$$\left(\int_{T_\infty} \tilde{\varphi}_t(\tilde{A}_t)d\mu(t)\right)^r \leq \int_{T_\infty} \tilde{\varphi}_t(\tilde{A}_t^r)d\mu(t),$$

namely,

$$\left(\int_{T_\infty} P_t\varphi_t(\tilde{A}_t)d\mu(t)\right)^r \leq Q^{r-1}\int_{T_\infty} P_t\varphi_t(\tilde{A}_t^r)d\mu(t), \tag{2.2}$$

for all bounded continuous fields $(\tilde{A}_t)_{t\in T_\infty}$.
Put $\tilde{A}_t = A_t/P_t$ $(t \in T)$ and $\tilde{A}_\infty = 0$ in (2.2) to obtain

$$\left(\int_T \varphi_t(A_t)d\mu(t)\right)^r \leq Q^{r-1}\int_T P_t^{1-r}\varphi_t(A_t^r)d\mu(t)\,. \tag{2.3}$$

By the definitions of P_t and Q, (2.3) becomes (2.1). □

Remark 2.3. *Using analogous argument as in the proof of Theorem 2.2, one can prove the following Jensen's inequality. Suppose that* (φ_t) *is a continuous field of positive linear mappings* $\varphi_t : \mathfrak{A} \to \mathfrak{B}$, $\mathfrak{A}$ *and* $\mathfrak{B}$ *are unital* C^**-algebras,* (A_t) *is a bounded continuous field of self-adjoint elements in* $\mathfrak{A}$ *with spectra in an interval* J *such that* $0 \in J$, (β_t) *is a continuous nonnegative function such that* $\int_T \beta_t d\mu(t) > 0$ *and*

$$\int_T \beta_t\varphi_t\left(I\right)d\mu(t) \leq \int_T \beta_t d\mu(t)I\,. \tag{2.4}$$

If f *is an operator convex function on an interval* J *such that* $f(0) \leq 0$*, then,*

$$f\left(\frac{1}{\int_T \beta_t d\mu(t)}\int_T \beta_t\varphi_t\left(A_t\right)d\mu(t)\right) \leq \frac{1}{\int_T \beta_t d\mu(t)}\int_T \beta_t\varphi_t\left(f\left(A_t\right)\right)d\mu(t). \tag{2.5}$$

If equality holds in (2.4), then it is not necessary to assume that $0 \in J$ *and* $f(0) \le 0$.

A discrete version of the theorem above is the following result obtained by taking $T = \{1, \cdots, n\}$.

Corollary 2.4. *Let* $1 < r \le 2$, $\alpha_i > 0$ $(i = 1, \cdots, n)$, $A_1, \cdots A_n$ *be positive operators acting on a Hilbert space* H *and* φ_i $(i = 1, \cdots, n)$ *be positive linear mappings on* $\mathbb{B}(\mathrm{H})$ *satisfying*

$$\sum_{i=1}^{n} \alpha_i^{1/(1-r)} \varphi_i(I) \le \sum_{i=1}^{n} \alpha_i^{1/(1-r)} I \,. \tag{2.6}$$

Then,

$$\left(\sum_{i=1}^{n} \varphi_i(A_i) \right)^r \le \left(\sum_{i=1}^{n} \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^{n} \alpha_i \varphi_i(A_i^r) \,.$$

By setting $\varphi_i(A) = X_i^* A X_i$ in Corollary 2.4, we find the following result.

Corollary 2.5. *Let* $1 < r \le 2$, $\alpha_i > 0$, *for* $i = 1, \cdots, n$, $A_1, \cdots A_n$ *be bounded operators acting on a Hilbert space* H *with* $A_i \ge 0$ *and* $X_1, \cdots, X_n \in \mathbb{B}(\mathrm{H})$ *satisfying*

$$\sum_{i=1}^{n} \alpha_i^{1/(1-r)} X_i^* X_i \le \sum_{i=1}^{n} \alpha_i^{1/(1-r)} I \,. \tag{2.7}$$

Then,

$$\left(\sum_{i=1}^{n} X_i^* A_i X_i \right)^r \le \left(\sum_{i=1}^{n} \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^{n} \alpha_i X_i^* A_i^r X_i \,.$$

Condition (2.7) trivially holds if $X_i^* X_i \le I$, for all $i = 1, \cdots, n$. In fact, we can give another proof of the result for this case.

Corollary 2.6. *Let* $A_1, \cdots A_n, X_1, \cdots, X_n \in \mathbb{B}(\mathrm{H})$ *with* $A_i \ge 0$, $X_i^* X_i \le I$, *for* $i = 1, \cdots, n$, *and* $1 < r \le 2$, $\alpha_i > 0$, *for* $i = 1, \cdots, n$. *Then,*

$$\left(\sum_{i=1}^{n} X_i^* A_i X_i \right)^r \le \left(\sum_{i=1}^{n} \alpha_i^{1/(1-r)} \right)^{r-1} \sum_{i=1}^{n} \alpha_i X_i^* A_i^r X_i \,.$$

Proof. First note that $0 \leq (X_i^* A_i X_i)^r \leq X_i^* A_i^r X_i$, for each i; cf. [4, Theorem 2.1]. For $i = 1, \cdots, n$, set $\beta_i = \alpha_i^{1/(1-r)}$ and $B_i = X_i^* A_i X_i / \beta_i$. We have,

$$\begin{aligned}
\left(\sum_{i=1}^n X_i^* A_i X_i\right)^r &= \left(\sum_{i=1}^n \beta_i B_i\right)^r \\
&= \left(\sum_{i=1}^n \beta_i \sum_{j=1}^n \frac{\beta_j}{\sum_{k=1}^n \beta_k} B_j\right)^r \\
&= \left(\sum_{i=1}^n \beta_i\right)^r \left(\sum_{i=1}^n \frac{\beta_i}{\sum_{k=1}^n \beta_k} B_i\right)^r \\
&\leq \left(\sum_{i=1}^n \beta_i\right)^r \frac{\sum_{i=1}^n \beta_i B_i^r}{\sum_{i=1}^n \beta_i} \\
&\qquad \text{(by the operator convexity of } f(t) = t^r\text{)} \\
&= \left(\sum_{i=1}^n \beta_i\right)^{r-1} \sum_{i=1}^n \beta_i^{1-r} (X_i^* A_i X_i)^r \\
&= \left(\sum_{i=1}^n \alpha_i^{1/(1-r)}\right)^{r-1} \sum_{i=1}^n \alpha_i (X_i^* A_i X_i)^r \\
&\leq \left(\sum_{i=1}^n \alpha_i^{1/(1-r)}\right)^{r-1} \sum_{i=1}^n \alpha_i X_i^* A_i^r X_i\,.
\end{aligned}$$

□

Proposition 2.7. *Let* $A_1, \cdots A_n \in \mathbb{B}(\mathrm{H})$ *with* $A_i^* A_j = 0$, *for* $1 \leq i \neq j \leq n$, *and* $2 < r \leq 4$, $\alpha_i > 0$, *for* $i = 1, \cdots, n$. *Then,*

$$\left|\sum_{i=1}^n A_i\right|^r \leq \left(\sum_{i=1}^n \alpha_i^{2/(2-r)}\right)^{(r-2)/2} \sum_{i=1}^n \alpha_i |A_i|^r$$

and

$$\left\|\sum_{i=1}^n A_i\right\|^r \leq \left(\sum_{i=1}^n \alpha_i^{2/(2-r)}\right)^{(r-2)/2} \sum_{i=1}^n \alpha_i \|A_i\|^r. \tag{2.8}$$

Proof.

$$\begin{aligned}
\left|\sum_{i=1}^{n} A_i\right|^r &= \left(\left|\sum_{i=1}^{n} A_i\right|^2\right)^{r/2} \\
&= \left(\sum_{i,j=1}^{n} A_i^* A_j\right)^{r/2} \\
&= \left(\sum_{i=1}^{n} |A_i|^2\right)^{r/2} \\
&\leq \left(\sum_{i=1}^{n} \alpha_i^{2/(2-r)}\right)^{(r-2)/2} \sum_{i=1}^{n} \alpha_i \left(|A_i|^2\right)^{r/2} \\
&\qquad \text{(by Corollary 2.6 with } X_i = I\text{)} \\
&= \left(\sum_{i=1}^{n} \alpha_i^{2/(2-r)}\right)^{(r-2)/2} \sum_{i=1}^{n} \alpha_i |A_i|^r .
\end{aligned}$$

Inequality (2.8) is easily deduced from the fact that $\|Z\|^r = \| \, |Z|^r \|$ for each $Z \in \mathbb{B}(\mathrm{H})$. □

Remark 2.8. *It is clear that* $A_1, \cdots A_n \in \mathbb{B}(\mathrm{H})$ *have orthogonal ranges if and only if* $A_i^* A_j = 0$. *An example of such operators is obtained by considering an orthogonal family* $(e_i)_{1 \leq i \leq n}$ *and a vector* x *in* H *and defining the rank one operators* $A_i : \mathrm{H} \to \mathrm{H}$ *by* $A_i = e_i \otimes x$, $1 \leq i \leq n$. *Then,* $A_i^* A_j = \langle e_j, e_i \rangle \, x \otimes x$, *for all* $1 \leq i, j \leq n$.

References

[1] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[2] W.S. Cheung and J.E. Pečarić, Bohr's inequalities for Hilbert space operators, *J. Math. Anal. Appl.* **323** (1) (2006) 403–412.

[3] F. Hansen and G.K. Pedersen, Jensen's operator inequality, *Bull. London Math. Soc.* **35** (2003) 553–564.

[4] F. Hansen and G.K. Pedersen, Jensen's inequality for operators and Löwner's theorem, *Math. Ann.* **258** (1982) 229–241.

[5] F. Hansen, J.E. Pečarić and I. Perić, *Jensen's operator inequality and its converses, Math. Scand.* **100** (1) (2007) 61–73.

[6] O. Hirzallah, Non-commutative operator Bohr inequality, *J. Math. Anal. Appl.* **282** (2003) 578–583.

[7] M.S. Moslehian and R. Rajić, Generalizations of Bohr's inequality in Hilbert C^*-modules, *Linear Multilinear Algebra* (to appear).
[8] V.I. Paulsen and D. Singh, Bohr's inequality for uniform algebras, *Proc. Amer. Math. Soc.* **132** (12) (2004) 3577–3579.
[9] J.E. Pečarić and Th.M. Rassias, Variations and generalizations of Bohr's inequality, *J. Math. Anal. Appl.* **174** (1) (1993) 138–146.
[10] M.P. Vasić and D.J. Kečkić, Some inequalities for complex numbers, *Math. Balkanica* **1** (1971) 282–286.
[11] F. Zhang, On the Bohr inequality of operators, *J. Math. Anal. Appl.* **333** (2007) 1264-1271.

M. S. Moslehian
Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.
And
Center of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, Iran.
Email: moslehian@ferdowsi.um.ac.ir and moslehian@ams.org

J. E. Pečarić
Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia.
Email: pecaric@mahazu.hazu.hr

I. Perić
Faculty of Food Technology and Biotechnology, Mathematics Department, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia.
Email: iperic@pbf.hr