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A VAN-KAMPEN THEOREM TYPE FOR HOMOTOPY GROUPS OF WEAK JOINS

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ABSTRACT. In this talk, we extend Mycielski's conjecture says that if X is a compact metric space, which is $(n-1)$-connected and locally $(n-1)$-connected, then $\pi_n(X)$ is either finitely generated or has the power of the continuum. Then, we try to extend the Van-Kampen theorem for weak joins, which is used to find the fundamental group of Hawaiian earring space, to higher homotopy groups.

1. Introduction and Motivation

In 1998, J. Pawlicowski [6] presented a forcing free proof of a conjecture of Mycielski [4] that the fundamental group of a connected locally connected compact metric space is either finitely generated or has the power of the continuum. In this talk, we show that Mycielski's conjecture is hold, also for higher homotopy groups.

J. W. Morgan and I. Morrison [2] among presenting a Van-Kampen theorem for weak joins proved that the canonical homomorphism $\phi : \pi_1(H) \to \varprojlim \pi_1(X_i)$ is injective, where H is the weak join of the X_i (as a special case H can be considered the Hawaiian earring). Here, we extend the injectivity of ϕ to higher homotopy groups with some conditions on the X_i.

2. extended Mycielski's conjecture

We recall a topological space X is called *n-semilocally simply connected at a point x* if there exists an open neighborhood U of x for which any n-loop in U is nullhomotopic in X. Moreover, X is said to be *n-semilocally simply connected* if

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it is n-semilocally simply connected at each point (see [1]). A space X is called *n-connected* for $n \geq 0$ if it is path connected and $\pi_k(X, x)$ is trivial for every base point $x \in X$ and $1 \leq k \leq n$. X is called *locally n-connected* if for each $x \in X$ and each neighborhood U of x, there is a neighborhood $V \subseteq U \subseteq X$ containing x so that $\pi_k(V) \longrightarrow \pi_k(U)$ is zero map for all $0 \leq k \leq n$ and for all basepoint in V.

Lemma 2.1. *Suppose X is an $(n-1)$-connected, locally $(n-1)$-connected compact metric space and $\pi_n(X)$ is not finitely generated. Then there exists $x \in X$ such that for each positive integer m, there exists an n-loop f_m at x with diameter $< 2^{-m}$ which is not nullhomotopic. In particular, X is not n-semilocally simply connected at x.*

Suppose that the n^{th}homotopy group of X is not finitely generated. Similar to [6], we define an equivalence relation $\{0,1\}^N$ via homotopic n-loops such as follows: First, take a point $x \in X$ and a sequens $\{f_m\}_{m\in N}$ of n-loops as claimed in Lemma 2.1. For each $\alpha \in \{0,1\}^N$, let f_m^α be the constant n-loop at x, if $\alpha(m) = 0$, otherwise let $f_m^\alpha = f_m$. Define an n-loop f_α at x as $f_0^\alpha * f_1^\alpha * \ldots$. Write $\alpha \approx \beta$ if $f_\alpha \sim f_\beta$. Then $\approx$ is an equivalence relation and it has continuum many equivalence classes; that is, there is a set of size of continuum of mutually non-homotopic n-loops. Therefore, we have the following theorem which is the extension of *Mycielski's conjecture* to higher homotopy groups.

Theorem 2.2. *Suppose X is a compact metric space, which is $(n-1)$-connected, locally $(n-1)$-connected. Then $\pi_n(X)$ is either finitely generated or has the power of the continuum.*

3. A Van-Kampen theorem type for homotopy groups of weak joins

In this section, we intend to extend the van-Kampen theorem for weak joins, which is used to find the fundamental group of Hawaiian earring space (see [2]), to higher homotopy groups. As a suitable model, consider the subspace H of R^{n+1} consisting of the infinite family of n-spheres C_i, where $n \geq 2$ and $i \in N$, of radius $\frac{1}{i}$ and center $(\frac{1}{i}, 0, \cdots, 0)$, mutually tangent to n-plane $x_1 = 0$ at the origin. We define X_i to be join of C_j's, for $1 \leq j \leq i$; and specially $X = \varprojlim X_j$.

We can assume that C_i, for $i \in N$, to be an arbitrary space which is $(n-1)$-connected, locally $(n-1)$-connected, n-semilocally simply connected, compact metric space.

It is easy to see that the groups $G_i = *_{j\leq i}\pi_n(X_j)$ form an inverse system of nested groups whose inverse limit denoted by G. Elements of G_i are called reduced words in the letter of type $j \leq i$ and elements of G are sequences $g = (g_j)_{j\in N}$ in conditions of these satisfying suitable compatibility conditions. We can use the various product constructions on groups which is discussed in Section 2 of [2] to study homotopy groups.

First we need to extend the notion of geometrically finiteness to n-loops. Let (X, y) be a pointed space and $\alpha : (I^n, \dot{I^n}) \to (X, y)$ be an n-loop so that $\alpha^{-1}(y) = \dot{I^n}$ and $\alpha^{-1}(X\backslash y) \bigcap \dot{I^n} = \emptyset$, where $\dot{I^n}$ is the boundary of unit n-cube I^n. An n-loop p in X is called *simple* if p is homotopic to α. We say that an n-loop p is *geometrically finite* if it is homotopic to a finite product of simple n-loops, i.e. there are simple n-loops $\alpha_1, \cdots, \alpha_k$ in such a way that $p \approx \alpha_1 * \cdots \alpha_k$.

The following assertion obtains from Lemma 2.1.

Corollary 3.1. *Suppose X is $(n-1)$-connected, locally $(n-1)$-connected, n-semilocally simply connected, compact metric space. Then each n-loop based at y is geometrically finite.*

We recall the following notion of [5]:

Given an n-loop α based at y in X, then any other n-loop γ based at y in X, with $\gamma \approx \alpha \quad (rel\ \dot{I^n})$ and $\gamma(I^n - I^n_1) = \{y\}$, is called a *concentration* of α on the subcube I^n_1.

Now suppose that p is a geometrically finite n-loop at y and $p \approx \alpha_1 * \cdots \alpha_k$, where α_i's are simple n-loops at y. Then by Lemma 2.5.2 of [5], there are subcubes $I^n_1, \cdots, I^n_k$ of I^n, given by $I^n_i = [a_{i1}, b_{i1}] \times [0,1] \times \cdots \times [0,1]$, $i = 1, \cdots, k$, where $0 \le a_{i1} \le b_{i1} \le 1$ and $b_{i1} \le a_{(i+1)1}$, $i = 1, \cdots, k-1$ (so that I^n_i is to the left of I^n_{i+1}) and concentrations γ_i of α_i on I^n_i in such a way that $p \approx \gamma_1 * \cdots * \gamma_k$. We denote the n-loop $\gamma_1 * \cdots * \gamma_k$ by q. Let $\pi_1 : R^n \to R$, defined by $\pi_1(x_1, \cdots, x_n) = x_1$. Then $\pi_1(q^{-1}(X\backslash y))$ is the union of a collection of disjoint open intervals which is denoted by W_p. The set W_p has a natural order induced by the ordering of $(0, 1)$.

Note that any n-loop p in H determines a sequence of n-loops p_i in X_i by $p_i = \pi_i \circ p$, where $\pi_i : H \to X_i$ is the natural projection, $i \in I$.

We say that p is *locally geometrically finite n-loop* if each p_i is geometrically finite n-loop. So we obtain the following lemma.

Lemma 3.2. *If each (X_i, y_i) is locally $(n-1)$-connected, $(n-1)$-connected, n-semilocally simply connected, compact metric pointed space, then every class in $\pi_n(H)$ is represented by a locally geometrically finite n-loop.*

In the sequel, let $\{(X_i, y_i)\}$ be a sequence of $(n-1)$-connected, locally $(n-1)$-connected, n-semilocally simply connected, compact metric spaces. Also, let $\mathrm{G}_i = *_{j \le i} \pi_n(X_j, y_j)$ and $G = \varprojlim \mathrm{G}_i$. The maps $\pi^i_j : X_i \to X_j$, $j < i$, are projections which induce homomorphisms $\pi^i_{j_*} : \pi_n(X_i) \to \pi_n(X_j)$. The corresponding homomorphisms $\pi_{i*} : \pi_n(H) \to \pi_n(X_i)$ are compatible with $\pi^i_{j_*}$ and imply the universal homomorphism $\phi : \pi_n(H) \to G = \varprojlim \pi_n(X_i)$. Now, the following result obtains.

Theorem 3.3. *The homomorphism ϕ is an injection whose image is the subgroup of locally eventually constant sequences which is isomorphic to $\pi_n(H)$.*

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