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# $n$-HOMOTOPICALLY HAUSDORFF SPACES

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ABSTRACT. In this talk, $n$-homotopically Hausdorff and strongly $n$-homotopically Hausdorff spaces are introduced. It is proved that every subset of 3-dimensional Euclidean space is 2-homotopically Hausdorff, and that every strongly $n$-homotopically Hausdorff space is $n$-homotopically Hausdorff. Moreover, some conditions are given for metric spaces to be $n$-homotopically Hausdorff or strongly $n$-homotopically Hausdorff at a point.

## 1. Introduction and Motivation

A space $X$ is homotopically Hausdorff at a point $x_0 \in X$, if for all nontrivial $\gamma \in \pi_1(X, x_0)$ there exists a neighborhood $U$ of $x_0$ such that no loop in $U$ is homotopic (in $X$) to $\gamma$ rel $x_0$. Furthermore, $X$ is homotopically Hausdorff, if it is homotopically Hausdorff at every point.

Homotopically Hausdorff spaces were first introduced by Cannon and Conner in 2006 ([1]). In [1] it is noted that the name homotopically Hausdorff is motivated by the fact that the path space $\Omega(X, x_0)$ is Hausdorff if and only if $X$ is homotopically Hausdorff at $x_0$.

In [3] the property of being homotopically Hausdorff is described and it is proved that every planar set is homotopically Hausdorff. In addition, Conner and Lamoreaux showed that if $X$ is a topological space which is first countable, homotopically Hausdorff, but it is not semilically simply connected, then $\pi_1(X)$ is uncountable ([3]). After that, Fischer and Zastrow proved the same theorem, but in a different and easier approach ([5]).

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In this talk, we describe the notion $n$-homotopically Hausdorffness, and extend the above result for higher homotopy groups. Moreover, we prove that every subset of 3-dimensional Euclidean space is 2-homotopically Hausdorff.

Recently, Conner and the others introduced strongly homotopically Hausdorff spaces, and gave some conditions for metric spaces which implies homotopically Hausdorff and strongly homotopically Hausdorff at a point, which were be easier to check for the spaces ([4]).

Here, we extend the notion of being strongly homotopically Hausdorff, and give the same conditions, which will be easier to check for metric spaces to being $n$-homotopically Hausdorff and strongly $n$-homotopically Hausdorff. Moreover, we show that every strongly $n$-homotopically Hausdorff space is $n$-homotopically Hausdorff.

## 2. $n$-Homotopically Hausdorff spaces

**Definition 2.1.** A space $X$ is called *$n$-homotopically Hausdorff* at $x_0 \in X$, if for any essential n-loop $\alpha$, based at $x_0$, there is an open neighborhood $U$ of $x_0$ for which $\alpha$ is not homotopic (rel $\dot{I}^n$) to an n-loop lying entirely in $U$.

$X$ is said to be *$n$-homotopically Hausdorff*, if it is $n$-homotopically Hausdorff at each of its points.

**Lemma 2.2.** *Let $X$ be a subset of $E^3$ and $N$ a closed disk in $E^3$ whose boundary is not contained in $X$. Let $\alpha_1$ and $\alpha_2$ be 2-loops in $X \cap int(N)$ based at $x_0$ which are homoyopic in $X$. Then there is a homotopy $F$ between $\alpha_1$ and $\alpha_2$ whose image is contained in $X \cap N$.*

**Theorem 2.3.** *Every subset of $E^3$ is 2-homotopically Hausdorff.*

*Proof.* Let $x_0 \in X \subseteq E^3$. Let $\alpha_0$ be a 2-loop in $X$ based at $X_0$ so that given any open set $U$ containing $X_0$, $\alpha_0$ is homotopic (in $X$ rel $X_0$) to a 2-loop lying entirely in $U$.

If $X_0$ is interior to $X$, then $\alpha_0$ is homotopic to a 2-loop whose image is in an open set $U \subseteq X$ which is homeomorphic to a Euclidean 3-dimensional disk, and thus $\alpha_0$ is nullhomotopic.

If $X_0$ is not interior to $X$, then there is a sequence of points in $E^3 - X$ which converges to $x_0$. If this is the case, let $p_0$ be a point in $E^3 - X$, and for each $n \in \mathbb{N}$, pick a point $p_n \in E^3 - X$ so that distance between $p_n$ and $x_0$ is no more that the minimum of $\frac{1}{n}$ and one-half the distance between $p_{n-1}$ and $x_0$ (i.e. $p_n \in B_{x_0}(min\{\frac{1}{n}, \frac{1}{2}d(x_0, p_{n-1})\}) \cap (E^3 - X)$).

Let $\epsilon_n = d(x_0, p_n)$. Choose a 2-loop $\alpha_n \subseteq B_{x_0}(\epsilon_n)$ based at $x_0$ which is homotopic to $\alpha_0$ (and hence to $\alpha_{n-1}$). Note that $\alpha_{n-1} \cup \alpha_n \subseteq B_{x_0}(\epsilon_{n-1})$ and that the boundary of $B_{x_0}(\epsilon_n)$ is a simple 2-loop containing the point $p_n$. Applying Lemma 2.2, we may choose a homotopy $F_n$ between $\alpha_n$ and $\alpha_{n-1}$ so that $F_n\ |_{I^2\times\{1\}}$ is $\alpha_n$, $F_n\ |_{I^2\times\{0\}}$ and the image of $F_n$ is contained in the closure of $B_{x_0}(\epsilon_{n-1})$. We sequentially adjoin the homotopies $F_i$ to form a homotopy $F$ by defining $F(x,y) = F_n(x, 2^{n+1}y - 1$ when $x \in I^2$ and $2^{-(n+1)} \leq y \leq 2^{-n}$, and defining $F(x,0) = x_0$. We show that $F$ is continuous.

*Case 1*: If $(x,y) \in I^3$ and $y > 0$, then continuity at $(x,y)$ follows from then continuity of at most two of the functions $F_{n-1}$ and $F_n$.

*Case 2*: If $(x,y) \in I^3$ and $y = 0$, then $F(x,y) = x_0$. Given any $\epsilon > 0$, we may choose a $k$ so that $\epsilon_k < \epsilon$. Now, for any $n > k$, the image of $F_n$ is contained in $B_{x_0}(\epsilon_n)$ and thus is a subset of $B_{x_0}(\epsilon_k)$. It follows any point in $B_{(x,y)}(2^{-(k+1)})$ would map to a point within $\epsilon_k$ and hence within $\epsilon$ of $x_0$.

Thus the 2-loop $\alpha_0$ is nullhomotopic and thus the set $X$ is 2-homotopically Hausdorff. □

Here, we give a condition for metric spaces which implies n-homotopically Hausdorfness at a point, which will be easier to check for our spaces. The basic idea is that for every small nullhomotopic n-loop, there is a nullhomotopy of small diameter.

**Theorem 2.4.** *Let $X$ be a metric space, and let $x_0 \in X$. Suppose $X$ has the property that for every $\epsilon > 0$ there is $\delta > 0$ such that for every map $f : B^{n+1} \to X$ with $f(S^n) \subseteq B_{x_0}(\epsilon)$, there is a map $g : B^{n+1} \to X$ such that $g\ |_{S^n} = f\ |_{S^n}$ and $g(B^{n+1}) \subseteq B_{x_0}(\epsilon)$. Then $X$ is n-homotopically Hausdorff at $x_0$.*

We recall a topological space $X$ is called *$n$-semilocally simply connected at a point $x$* if there exists an open neighborhood $U$ of $x$ for which any $n$-loop in $U$ is nullhomotopic in $X$. Moreover, $X$ is said to be *$n$-semilocally simply connected* if it is $n$-semilocally simply connected at each point (see [7]).

**Theorem 2.5.** *Suppose that $X$ has a countable Basis at $x_0$, that $X$ is $n$-homotopically Hausdorff at $x_0$, and that $X$ is not $n$-semilocally simply connected at $x_0$. Then $\pi_n(X, x_0)$ is uncountable.*

**Definition 2.6.** A space $X$ is called *strongly $n$-homotopically Hausdorff* at $x_0 \in X$, if for each essential n-loop $\gamma$ in $X$, there is an open neighborhood of $x_0$ that contains no n-loop freely homotopic (in $X$) to $\gamma$.

A compact space $X$ is said to be *strongly $n$-homotopically Hausdorff*, if it is strongly $n$-homotopically Hausdorff at each of its points.

**Theorem 2.7.** *If $X$ is strongly $n$-homotopically Hausdorff at $x_0 \in X$, then $X$ is $n$-homotopically Hausdorff at $x_0$.*

*Proof.* Let $\gamma$ be an n-loop based at $x_0$ that can be homotoped rel $x_o$ into arbitrarily small neighborhood of $x_0$ in $X$. Then since based pointed homotopies are a spacific type of (free) homotopy, we see that since $X$ is strongly $n$-homotopically Hausdorff at $x_0$, $\gamma$ must be nullhomotopic, and therefore $X$ is $n$-homotopically Hausdorff at $x_0$. □

Finally, we give a sufficient condition for being strongly $n$-homotopically Hausdorff at a point, which essentially says that for every pair of homotopic n-loops nearby a point, there is a homotopy of small diameter between them.

**Theorem 2.8.** *Let $X$ be a compact metric space and $x_0 \in X$ such that for every $\epsilon > 0$ there is a $\delta > 0$ such that for every map $f : S^n \times [0,1] \to X$ such that $f|_{S^n\times\{0\}}$ is not freely nullhomotopic, and $f(S^n \times \{0,1\}) \subseteq B_{x_0}(\delta)$, there is a map $g : S^n \times [0,1] \to X$ such that $g|_{S^n\times\{0,1\}} = f|_{S^n\times\{0,1\}}$, and $g(S^n \times [0,1]) \subseteq B_{x_0}(\epsilon)$. Then $X$ is strongly $n$-homotopically Hausdorff at $x_0$.*

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