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The Optimal Method for Solving Continuous Linear and Nonlinear Programs

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Abstract

In this paper, we consider a class of continuous nonlinear programs posed in a function space called separated continuous nonlinear programs (SCNP). In first step we transform the constraints of problem to integral constraints. Then by using Riesz representation theorem we reduce the later problem to an infinite-dimensional linear programming problem. After that we approximate the later problem to a finite-dimensional linear programming problem. Then by solving this problem, we obtain an approximate solution for original problem in the form piecewise constant function. Finally, we investigate several numerical examples.

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1- Introduction

In 1953, Bellman introduced a class of optimization problems which he called bottleneck problems. These problems have now become known as continuous linear programs because they can be formulated as linear programs having variables that are functions of time as follows:

$$\textbf{CLP}: \max \int_0^T c(t)^T x(t)dt$$

Subject to

$$B(t)x(t) + \int_0^T K(s,t)x(s)ds \le b(t)$$

$$x(t) \ge 0, t \in [0,T]$$

with x(t), c(t) and the elements of B(t) and K(s,t) bounded measurable functions. In 1993, Pullan represented an algorithm for solving a class of continuous linear programs in a paper [6] (it's notable that he has solved weak version of problem). In 1994, Anderson and Philpott have worked on this problem [1]. Then in 1995, Pullan analyzed the problem in the existence of solution [7].

In this paper, we reduce the condition of the problem, and then solve the nonlinear form of problem with measure theory (See [8]). So we investigate a generalization form of the subclass of continuous nonlinear programs called separated continuous nonlinear programs (SCNP) .

$$\textbf{SCNP}: \min \int_0^T c(t)^T F(x(t))dt$$

Subject to

$$\int_0^T G(s,x(s))ds + y(t) = a(t) \tag{1}$$

$$H(t,x(t)) \le b(t), t \in [0,T]$$

Here x(t), b(t) and c(t) are bounded measurable functions and y(t) and a(t) are absolutely continuous functions. The dimensions of x(t) and y(t) are n and m, respectively. Note that the F, G, H are nonlinear functions.

2. Formulation of the Problem

SCNP has an alternative formulation optimal control problem. Now we consider a general form as follows:

$$\min \quad I[x,y] = \int_{t_a}^{t_b} f_0(t,y(t),x(t))dt \tag{2a}$$

Subject to

$$\dot{y} = g(t,y(t),x(t)) \qquad \text{a.e.} \qquad t \in [t_a,t_b] \tag{2b}$$

$$h(t,y(t),x(t)) \le 0 \qquad \text{a.e.} \qquad t \in [t_a,t_b] \tag{2c}$$

Let $\Omega = J \times A \times U$, where $J = [t_a, t_b]$ and A, U are the compact sets of IR^n, IR^m, respectively, f: $\Omega \to IR$, g : $\Omega \to IR^n$, and h : $\Omega \to IR^k$ are continuous functions. Here the trajectory function $y(t) \in U$ is lebesgue-measurable.

3. Classical Control problems

We shall say that a trajectory-control pair p = [y(.),x(.)], is admissible if the following conditions hold:

i) $y(t) \in A$, $t \in J$.

ii) $x(t) \in U$, $t \in J$.

iii) The boundary conditions $y(t_a) = y_a$ and $y(t_b) = y_b$ is satisfied, where y_b is unknown.

iv) The pair p satisfies the differential equation (2b) and inequality (2c) a.e. on J^0.

Let B an open ball in IR^{n+1} containing $J \times A$. Let C^1 (B) be the space of all real-valued continuously differentiable functions on B which are bounded on B together with their first derivatives. Let $\phi \in C^1$ (B), and define function ϕ^g as follows:

$$\phi^g(t, y, x) = \phi_y(t, y)g(t, y, x) + \phi_t(t, y) \qquad (t, y, x) \in \Omega \qquad (3.1)$$

The function ϕ^g is in the space $C(\Omega)$ of all real-valued continuous functions defined on the compact set Ω. Thus we have

$$\int_J \phi^g(t, y(t), x(t))dt = \int_J [\phi_y(t, y)\dot{y} + \phi_t(t, y)]dt = \int_J \dot{\phi}(t, y)dt$$

$$= \phi(t_b, y_b) - \phi(t_a, y_a) \equiv \Delta\phi, \qquad \phi \in C^1(B) \qquad (3.2)$$

Let us now consider a weak version of (2b). Let $D(J^0)$ be the space of infinitely differentiable real-valued functions with compact support in J^0. Define

$$\psi_j(t, y, x) = y_j\psi'(t) = g_j(t, y, x)\psi(t) \qquad j = 1,2,...,n \qquad \psi \in D(J^0). \quad (3.3)$$

Then

$$\int_J \psi_j(t, y, x)dt = y_j\psi(t)\big|_J - \int_J [\dot{y}_j(t) - g_j(t, y, x)]\psi(t)dt = 0$$

Put

$$\phi(t, y) = \theta(t) \qquad (t, y, x) \in \Omega \qquad (3.4)$$

that is, a function which depends on the time variable only; then $\phi^g(t,y,x) = \dot{\theta}(t)$. If p is an admissible pair, the equality (3.2) with the choice (3.4) for the function ϕ implies that

$$\int_J F(t, y, x)dt = a_f, \qquad f \in C_1(\Omega)$$

Where a_f is the integral of f over J, independent of y and x; we have put f for $\dot{\theta}$. ($C_1(\Omega)$ is the subspace of the $C(\Omega)$, the functions in this space which depend only on the t). Also if p = [y(.), x(.)] satisfies inequality (2c) then

$$\int_J \big| h_i(t, y(t), x(t)) + |h_i(t, y(t), x(t))| \big| dt = 0 \qquad (3.5)$$

Where h_i is the ith component of h. Now the classical problem is transformed to minimization of the following functional

$$I[y(.),x(.)] = \int_J f_0(t,y(t),x(t))dt \tag{3.6a}$$

subject to

$$\int_J \phi^g(t,y(t),x(t))dt = \Delta\phi, \qquad \phi \in C'(B) \tag{3.6b}$$

$$\int_J \psi_j(t,y(t),x(t))dt = 0, \qquad j = 1,2,...,n, \quad \psi \in D(J^0) \tag{3.6c}$$

$$\int_J f(t,y(t),x(t))dt = a_f, \qquad f \in C_1(\Omega) \tag{3.6d}$$

$$\int_J |h_i(t,y(t),x(t)) + |h_i(t,y(t),x(t))||dt = 0, \qquad i = 1,2,...,k \tag{3.6e}$$

Now,

(1) The mapping

$$\Lambda_p : F \to \int_\Omega F(t,y(t),x(t))dt, \qquad F \in C(\Omega),$$

Defines a positive, linear functional on $C(\Omega)$

(2) By the Riesz representation theorem, there exists a unique positive Radon measure μ on Ω which

$$\Lambda_p(F) = \int_\Omega F(t,y,x)dt = \int_\Omega F d\mu \equiv \mu(F), \qquad F \in C(\Omega)$$

Let $M^+(\Omega)$ be the set of all positive Radon measure on Ω. Define the positive Radon measure $\mu \in M^+(\Omega)$ such that minimizes the following linear functional

$$I(\mu) = \mu(f_0) \tag{3.7a}$$

subject to

$$\mu(\phi^g) = \Delta\phi, \qquad \psi \in C'(B), \tag{3.7b}$$

$$\mu(\psi_j) = 0, \qquad j = 1,2,...,n \qquad \psi \in D(J^0), \tag{3.7c}$$

$$\mu(f) = a_f, \qquad f \in C_1(\Omega), \tag{3.7d}$$

$$\mu(|h_i + |h_i||) = 0 \qquad i = 1,2,...,k. \tag{3.7e}$$

We define Q to be the set of all measures in $M^+(\Omega)$ that satisfy equalities (3.7b)-(3.7e) than we can show that there exists an optimal measure μ^* in the Q for which $\mu^*(f_0) \le \mu(f_0)$ for all $\mu \in Q$(see[8]).

This problem is an infinite-dimensional linear programming problem and all the functions in (3.7a)-(3.7e) are linear with respect to measure μ. We obtain the approximate solution of this problem by the solution of a finite-dimensional linear program.

4. The First Approximation

In this section, we are limiting constraint (3.7a)-(3.7e) as follows:

i) We choose the function $\phi(t,y)=y_i^r, r=1,2,,,,.M_1$ and i=1,2,..,n. Then we have $\phi^g = ry_i^{r-1}g_i(t,y,x)$.

ii) We consider

$$\psi_r(t)=\sin[\frac{2\pi r(t-t_a)}{\Delta t}], \qquad r=1,2,...,M_{21},$$

$$\psi_r(t)=1-\cos[\frac{2\pi r(t-t_a)}{\Delta t}], \qquad r=M_{21}+1,...,2M_{21},$$

Where $\Delta t = t_b - t_a, M_2 = 2nM_{21}$. We shall call χ_h, h=1,2,…, the sequence of functions of the type

$$\psi_r(t,y,x)=y_r\dot{\psi}(t)+g_r(t,y,x)\psi(t), \qquad r=1,2,...,M_2.$$

iii) For the third type of constraints, we choose

$$\theta_s=\begin{cases}1 & t\in J_s\\ 0 & \text{otherwise.}\end{cases}$$

with $J_s=[\frac{t_a+(s-1)\Delta t}{L},\frac{t_a+s\Delta t}{L}] \qquad s=1,2,....,L.$

5. The Second Approximation

In the previous section, we have limited the number of constraints in the original linear program; the underlying space is not, however, finite-dimensional.

Definition: Let A be borel set and z a point in the space Ω. The Radon measure δ_z will be called atomic measure if

$$\delta(z)(A)=\begin{cases}1 & z\in A\\ 0 & \text{otherwise.}\end{cases}$$

Now we approximate μ^* by a linear combination of atomic measure in following proposition of [8].

Proposition: Let ω be a countable dense subset of Ω. Given ε > 0, a measure $\nu\in M^+(\Omega)$ can be found such that

$$|(\mu^*-\nu)f_0|<\varepsilon,$$

and

$$|(\mu^*-\nu)\phi_j^g|<\varepsilon, \quad j=1,2,...,M,$$

$$|(\mu^*-\nu)|h_i+|h_i|||<\varepsilon, \quad i=1,2,...,k,$$

and the measure ν has the form

$$\nu=\sum_{k=1}^{M}\alpha_k\delta(z^k),$$

where $\delta(z^k)$ is atomic measure, $z^k\in\omega,\alpha_k\geq 0, \qquad k=1,2,...,M.$

The infinite-dimensional linear programming problem in (3.7) can be approximated by the following linear programming problem in which $z_j, j = 1,2,..., N$ belongs to a dense subset of Ω.

$$\min \quad \sum_{j=1}^{N} \alpha_j f_0(z_j), \tag{5.1a}$$

subject to

$$\sum_{j=1}^{N} \alpha_j \phi_i^g(z_j) = \Delta\phi_i, \qquad i = 1,2,..., M_1, \tag{5.1b}$$

$$\sum_{j=1}^{N} \alpha_j \chi_A(z_j) = 0, \qquad h = 1,2,..., M_2, \tag{5.1c}$$

$$\sum_{j=1}^{N} \alpha_j \theta_s(z_j) = a_f, \qquad s = 1,2,..., L, \tag{5.1d}$$

$$\sum_{j=1}^{N} \alpha_j (|h_i + |h_i||(z_j) = 0, \qquad i = 1,2,..., k, \tag{5.1e}$$

$$\alpha_j \geq 0, \qquad j = 1,2,..., N,$$

where $z_j = (t_j, y_j, x_j) \in \omega, j = 1,2,..., N$ are constructed by dividing the sets J, A, U into the number of equal subsets.

By using manner similar (see [8]) we can approximate the optimal pair [y(.),x(.)] by considering

$$\lambda_k = \sum_{j \leq k} \alpha_j$$

such that if we set $t \in (\lambda_{k-1}, \lambda_k)$ than $x(t) \approx x_k$ and the trajectory y(.) will be obtained by the equation (2b).

Of course, we note that we need only to control variable x(.).

6. Computational Results

We now present results from solving two simple problems. The first is continuous LP programs and the second is continuous NLP programs.

Example.6.1. Consider the following SCLP problem:

$$\min \int_0^1 c(t)x(t)dt$$

subject to

$$\int_0^t x(s)ds + y(t) = 1$$

$$0 \leq x(t) \leq 1, y(t) \geq 0, t \in [0,1]$$

where

$$c(t)=\begin{cases} t^2\sin\frac{1}{t} & t\in(0,1] \\ 0 & t=0 \end{cases}$$

This problem has optimal solution as follows:

$$x(t)=\begin{cases} 1 & \text{a.e on } S \\ 0 & \text{a.e on } [0,1]-S \end{cases}$$

where $S=\{t\in[0,1]:c(t)<0\}$. Now, we are using measure theory method for solving above problem. We consider the set $\Omega=[0,1]\times[0,1]\times[0,1]$. We divide the intervals J, A and U into 10 equal subinterval and so $N=10^3$, then we choose the points $z_r=(t_r,y_r,x_r)$ from the set Ω such that

$t_r=0.01(i-1)+0.1(i-1)$ $x_r=0.011(j-1)+0.1(j-1)$ $y_r=0.01(k-1)+0.1(k-1)$

$r=i+(j-1)10+(k-1)100$ and i, j and k =1,…,10. Further we choose $M_1=1, M_2=6, L=10$ and K=1. So we have the following linear programming problem.

$$\min \quad \sum_{j=1}^{1000}(t_j^2\sin\frac{x_i}{t_j})\alpha_j$$

subject to

$$\sum_{j=1}^{1000}x_j\alpha_j+\beta=1$$

$$\sum_{j=1}^{1000}[2\pi hy_i\cos(2\pi ht_j)-x_j\sin(2\pi ht_j)]\alpha_j=0\ ,\ h=1,2,3$$

$$\sum_{j=1}^{1000}[2\pi hy_i\cos(2\pi ht_j)-x_j(1-\cos(2\pi ht_j))]\alpha_j=0\ ,\ h=1,2,3$$

$$\alpha_{1+100(i-1)}+\alpha_{2+100(i-1)}\cdots+\alpha_{100+100(i-1)}=0.1, i=1,2,...,10$$

$$\sum_{j=1}^{1000}||x_j-1|+x_j-1|\alpha_j=0$$

$\alpha_j\geq 0, j=1,2,...,1000, \beta$ is free. By solving this problem, we obtain $\alpha_j, j=1,2,...,1000$, and β, and then we obtain the approximate solution x(.). In this example, the optimal value of the cost function is 0.0227. Graph of the approximate solution x(.) is shown in Fig.1.

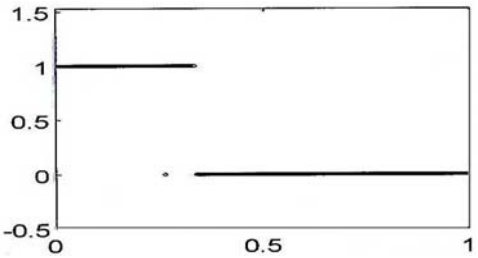

Fig.1. Approximated optimal solution x(.)

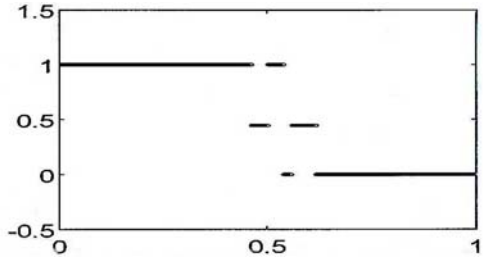

Fig.2. Approximated optima solution x(.)

Example.6.2. Consider the following SCNP problem:

$$\min \int_0^1 c(t)x^2(t)dt$$

Subject to

$$\int_0^t x(s)ds + y(t) = 1,$$

$$x^3(t) \le t^2 + 5,$$

$$x(t), y(t) \ge 0, \quad t \in [0,1]$$

where

$$c(t) = \begin{cases} t^3 \sin\frac{1}{t^2} & t \in (0,1] \\ 0 & t = 0. \end{cases}$$

Here

$$\Omega = [0,1] \times [0,1] \times [0,1].$$

Again we divide the intervals J, A and U into 10 equal subinterval and so $N = 10^3$, then we choose the points $z_r = (t_r, y_r, x_r)$ from the set Ω such that $t_r = 0.01(i-1) + 0.1(i-1)$ $x_r = 0.011(j-1) + 0.1(j-1)$ $y_r = 0.01(k-1) + 0.1(k-1)$ $r = i + (j-1)10 + (k-1)100$ and i,j and k=1,…,10. Also we choose $M_1 = 1, M_2 = 6, L = 10$ and $K = 1$.So we have the following linear programming problem.

$$\min \quad \sum_{j=1}^{1000} (t_j^3 \sin\frac{x_j^2}{t_j^2})\alpha_j$$

subject to

$$\sum_{j=1}^{1000} x_j \alpha_j + \beta = 1$$

$$\sum_{j=1}^{1000} [2\pi h y_j \cos(2\pi h t_j) - x_j \sin(2\pi h t_j)]\alpha_j = 0 \quad h = 1,2,3$$

$$\sum_{j=1}^{1000} [2\pi h y_i \sin(2\pi h t_j) - x_j(1 - \cos(2\pi h t_j))]\alpha_j = 0 \quad h = 1,2,3$$

$$\alpha_{1+100(i-1)} + \alpha_{2+100(i-1)} \cdots + \alpha_{100+100(i-1)} = 0.1, i = 1,2,...,10$$

$$\sum_{j=1}^{1000} \| x_j^3 - t_j^2 - 5| + x_j^3 - t_j^2 - 5| \alpha_j = 0$$

$\alpha_j \ge 0, j = 1,2,...,1000, \beta$ is free. Then after solving this problem, we obtain an approximate solution x(.). The optimal value of the cost function is 0.0469. We show it in Fig.2.

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